Isotropy Groups of Algebraic Theories

Jason Parker
Joint with Pieter Hofstra and Philip Scott

University of Ottawa

FMCS 2017
June 10, 2017

Classifying Toposes

Definition

A **geometric theory** T is a (possibly infinitary) first-order theory whose axioms can be written as sequents of formulae constructed from the connectives $\top$ (truth), $\wedge$ (conjunction), $\bot$ (falsity), $\bigvee$ (possibly infinitary disjunction), and $\exists$ (existential quantification).

Every geometric theory T has a classifying topos $\mathcal{B}(T)$ for T-models, defined as follows:

Definition

A **classifying topos** for T-models is a topos $\mathcal{B}(T)$ with the property that for any cocomplete topos $\mathcal{E}$, there is an equivalence of categories

$$c_{\mathcal{E}} : \underline{Mod}(\mathcal{E}, T) \xrightarrow{\sim} Hom(\mathcal{E}, \mathcal{B}(T))$$

that is natural in $\mathcal{E}$.

Classifying Toposes

Given a geometric theory T, there is a **generic or universal model** U_T in the classifying topos $\mathcal{B}(T)$, which has the following **universal property**:

For any cocomplete topos $\mathcal{E}$ and T-model $\mathcal{M}$ in $\mathcal{E}$, $\exists!$ geometric morphism (up to isomorphism) $f : \mathcal{E} \to \mathcal{B}(T)$ such that

$$f^*(U_T) \cong \mathcal{M},$$

where $f^* : \mathcal{B}(T) \to \mathcal{E}$.

Classifying Toposes

Also, if $\mathcal{E}$ is any Grothendieck topos, then there is a unique geometric theory T (up to equivalence) such that $\mathcal{E} = \mathcal{B}(T)$. So every topos is classified by some geometric theory.

Isotropy Groups of Geometric Theories

If T is a geometric theory with classifying topos $\mathcal{B}(T)$ and universal model $U_T \in \mathcal{B}(T)$, then we can define the internal group object

$$(Aut(U_T), comp, inv, unit)$$

of all T-model automorphisms of the universal model U_T using the internal logic of the topos $\mathcal{B}(T)$ and the fact that $\mathcal{B}(T)$ has a small set of generators.

Isotropy Groups of Geometric Theories

Then we have the following:

Definition

Let $\mathcal{E}$ be a topos classified by the geometric theory T, so that $\mathcal{E} = \mathcal{B}(T)$ and $U_T \in \mathcal{B}(T) = \mathcal{E}$ is the universal model of T. Then we define the **isotropy group** $Z_{\mathcal{E}}$ of the topos $\mathcal{E}$ to be the internal group

$$Aut(U_T) \in \mathcal{B}(T) = \mathcal{E}$$

all T-automorphisms of U_T.

If T is a geometric theory, then by the isotropy group Z_T of T, we mean the isotropy group of the classifying topos $\mathcal{B}(T)$ of T.

Isotropy Groups of Algebraic Theories

Now let T be an algebraic or equational theory, whose signature contains only function symbols, and whose axioms are all equations between terms. We will analyze the structure of the isotropy group of T.

Recall that if T is an algebraic theory, then the classifying topos of T is the functor category

$$Sets^{fpTmod},$$

where $fpTmod$ is the category of finitely presented T-models in $Sets$.

Remark

If we regard T as a Lawvere theory, then $fpTmod^{op}$ is the finite limit completion of the Lawvere theory T.

Isotropy Groups of Algebraic Theories

If T is an algebraic theory with universal model $U_T \in Sets^{fpTmod}$, then

$$U_T : fpTmod \to Sets$$

acts on objects $M \in fpTmod$ by

$$U_T(M) = |M|,$$

where $|M|$ is the underlying set of M.

Then $U_T : fpTmod \to Sets$ factors through the category $Tmod$ of all T-models in Sets, so that U_T gives a T-model in $Sets^{fpTmod}$.

Isotropy Groups of Algebraic Theories

Note that since $Z_T \in Sets^{fpTmod}$ is an internal group of $Sets^{fpTmod}$, we have that $Z_T : fpTmod \to Grp$, and we will calculate $Z_T(M) \in Grp$ for any $M \in fpTmod$.

For our first characterization of the isotropy group of an algebraic theory T, first let $M \in fpTmod$ be arbitrary, where we define the forgetful functor

$$V_M : M \downarrow fpTmod \to fpTmod,$$

where $M \downarrow fpTmod$ is the category of all morphisms in $fpTmod$ with domain M.

Then we can consider the usual group $Aut(V_M)$ of all natural automorphisms of the functor V_M.

First Characterization

Then our first characterization of the isotropy group of an algebraic theory is given by the following:

Theorem

Let T be an algebraic theory with classifying topos $Sets^{fpTmod}$ and isotropy group $Z_T \in Sets^{fpTmod}$. Then $\forall M \in fpTmod$, there is a group isomorphism

$$Z_T(M) \cong Aut(V_M),$$

natural in M.

Coherent Systems of Automorphisms

If $M \in fpTmod$, then automorphisms of V_M can be described more concretely in terms of the following notion.

Definition

A **coherent system of automorphisms** of the model M in the category $fpTmod$ is given by a T-automorphism

$$\alpha : M \xrightarrow{\sim} M$$

such that $\forall$ morphism $f : M \to N$ in $fpTmod$, there is an automorphism

$$\alpha_f : N \xrightarrow{\sim} N$$

such that $\alpha_{id_M} = \alpha$ and for any commuting triangle

Definition Cont.

$$\begin{array}{ccc} M & \xrightarrow{f_1} & N_1 \\ & \searrow^{f_2} & \downarrow h \\ & & N_2 \end{array}$$

in *fpTmod*, we have that the following diagram commutes:

$$\begin{array}{ccc} N_1 & \xrightarrow{\alpha_{f_1}} & N_1 \\ h \downarrow & & \downarrow h \\ N_2 & \xrightarrow[\alpha_{f_2}]{} & N_2 \end{array}$$

Coherent Systems of Automorphisms

Then it follows directly from the definitions that a coherent system of automorphisms of a model $M \in fpTmod$ is **exactly** an automorphism of the forgetful functor $V_M : M \downarrow fpTmod \to fpTmod$.

For any model $M \in fpTmod$, we can then consider the group $CohSAut(M)$ of all coherent systems of automorphisms of M. Then we have

Corollary

Let T be an algebraic theory with classifying topos $Sets^{fpTmod}$ and isotropy group $Z_T \in Sets^{fpTmod}$. Then $\forall M \in fpTmod$, there is a group isomorphism

$$Z_T(M) \cong Aut(V_M) \cong CohSAut(M),$$

natural in M.

Second Characterization

We now wish to move from this first abstract characterization of the isotropy group Z_T, in terms of coherent systems of automorphisms, to a more syntactical and computation-friendly characterization.

Bergman considered coherent systems of automorphisms in the setting of groups, and proved the following result:

Theorem (Bergman)

If G is a group, then $G \cong CohSAut(G)$.

Equivalently, Bergman proved that the universal group is complete, where a group G is complete if the canonical map $G \to Aut(G)$ is an isomorphism, where $g \mapsto i_g$ and $i_g \in Aut(G)$ is given by $i_g(h) = ghg^{-1}$ $\forall h \in G$.

Second Characterization

The proof of this theorem involved some techniques which then inspired our second characterization of the isotropy group of an algebraic theory T.

First, if $M \in fpTmod$, then $M\langle x \rangle$ is the T-model gained by adjoining an indeterminate x to M.

Then an element $[t(x)] \in M\langle x \rangle$ is (an equivalence class of) a word in x, the elements of M, and the function symbols of T, modulo the equations of T and relations among elements of M.

Second Characterization

If $[t(x)]$ is any element of $M\langle x \rangle$, then $[t(x)]$ induces a function

$$[t(x)]^M : M \to M.$$

And if $h : M \to N$ is any morphism of $fpTmod$, then h induces a unique map

$$h_x : M\langle x \rangle \to N\langle x \rangle$$

in the obvious way.

We can also view $M\langle x \rangle$ as a monoid, under the operation of substitution and with $[x]$ as the identity element. Then the above process yields a monoid homomorphism

$$M\langle x \rangle \to End(M),$$

where $End(M)$ is the monoid of endomaps of M.

Second Characterization

Now we make the following definition:

Definition

Let T be an algebraic theory. For any model $M \in fpTmod$, let $\mathbf{M}^{\mathbf{coh}}\langle \mathbf{x} \rangle$ be the set of all $[t(x)] \in M\langle x \rangle$ such that for any morphism $h : M \to N$ in $fpTmod$, the induced function

$$h_x([t(x)])^N : N \to N$$

is a T-automorphism of N.

Second Characterization

Then $M^{coh}\langle x\rangle$ is a group (as a submonoid of $M\langle x\rangle$), and we have

Theorem

Let T be an algebraic theory with classifying topos $Sets^{fpTmod}$ and isotropy group $Z_T \in Sets^{fpTmod}$. Then $\forall M \in fpTmod$ there is a group isomorphism

$$Z_T(M) \cong M^{coh}\langle x\rangle,$$

natural in M.

So the isotropy group of T sends every $M \in fpTmod$ to the group of all elements $[t(x)]$ in $M\langle x\rangle$ whose homomorphic images always induce T-automorphisms.

Third Characterization

We now want to determine if there is a condition on elements $[t(x)] \in M\langle x\rangle$ that will characterize the sets $M^{coh}\langle x\rangle$ without quantifying over arbitrary homomorphisms out of M. It turns out that there is such a condition, which we describe as follows.

First, note that an element $[t(x)]$ in the monoid $M\langle x\rangle$ is invertible iff there is an element $[t^{-1}(x)] \in M\langle x\rangle$ such that

$$[t[t^{-1}/x]] = [x] = [t^{-1}[t/x]].$$

Third Characterization

Definition

Let T be an algebraic theory with $M \in fpTmod$. Let $M^{hom}\langle x \rangle$ be the set of all $[t(x)] \in M\langle x \rangle$ such that:

1. $[t(x)]$ is invertible in the monoid $M\langle x \rangle$.
2. For any n-ary function symbol f of T ($n \geq 0$) and indeterminates $x_1, \ldots, x_n$, the equality

$$[f(t(x_1), \ldots, t(x_n))] = [t(f(x_1, \ldots, x_n))]$$

holds in the T-model $M\langle x_1, \ldots, x_n \rangle$. We say that $[t(x)]$ **commutes generically** with all operations of T.

It is easy to prove that if $[t(x)] \in M^{hom}\langle x \rangle$, then $t(x)$ has at least one occurrence of x.

Third Characterization

Now we have the third and most computation-friendly characterization of the isotropy group of an algebraic theory T:

Theorem

Let T be an algebraic theory with $M \in fpTmod$. Then

$$M^{coh}\langle x\rangle = M^{hom}\langle x\rangle.$$

So if $Z_T \in Sets^{fpTmod}$ is the isotropy group of T, then $\forall M \in fpTmod$ we have

$$Z_T(M) \cong M^{hom}\langle x\rangle,$$

natural in M.

Examples

With this concrete characterization of the isotropy group of an algebraic theory $\mathbb{T}$ at hand, we can now (more easily) compute the isotropy groups of several algebraic theories $\mathbb{T}$:

- If $\mathbb{T}$ has no axioms, then the isotropy group of $\mathbb{T}$ is trivial, i.e. $\forall M \in \mathbb{T}\text{-}\mathit{mod}$ we have $\mathcal{Z}_{\mathbb{T}}(M) = \{[x]\} \cong 1$, the trivial group.
- If $\mathbb{T}$ is the theory of groups, then Bergman essentially proved that $\forall G \in \mathit{Group}$ we have

$$\mathcal{Z}_{\mathbb{T}}(G) = \{[gxg^{-1}] \in G\langle x\rangle \mid g \in G\} \cong G.$$

- If $\mathbb{T}$ is the theory of monoids, then $\forall M \in \mathit{Monoid}$ we have

$$\mathcal{Z}_{\mathbb{T}}(M) = \{[mxm'] \in M\langle x\rangle \mid m \text{ is invertible in } M \text{ and } m' = m^{-1}\}.$$

Examples

- If T is the theory of abelian groups, then $\forall G \in fpAb$ we have

$$Z_T(G) = \{[x], [-x]\} \cong \mathbb{Z}_2.$$

- If T is the theory of commutative monoids, then the isotropy group of T is trivial.
- If T is the theory of non-commutative rings with 1, then $\forall R \in fpRing$ we have

$$Z_T(R) = \{[rxr^{-1}] \in R\langle x\rangle \mid \; r \text{ is a unit } \}.$$

- If T is the theory of commutative rings with 1, then the isotropy group of T is trivial.
- If T is the theory of R-modules for some commutative ring R, then $\forall M \in fpRmod$, we have that

$$Z_T(M) = \{(0_M, rx) \in M\langle x\rangle \mid \; r \text{ is a unit } \}.$$

Examples

- Let T be the theory of a bijection, so that T has two unary operation symbols f and f^{-1} with the axioms

$$f(f^{-1}(x)) = x = f^{-1}(f(x)).$$

Then $\forall M \in fpTmod$, we have that

$$Z_T(M) = \{[f^n(x)] \in M\langle x\rangle \mid n \in \mathbb{Z}\} \cong \mathbb{Z},$$

where of course $[f^0(x)] = [x]$ and $[f^{-n}(x)] = [(f^{-1})^n(x)]$ if $n \geq 1$.

Final Characterization

Many of these examples suggest that the isotropy group of T has a close connection to the "inner automorphisms" of T-models.

There is a universal algebra definition of inner automorphism for any algebraic theory T (given by Csakany in 1965), which yields the usual notion for groups but **not** for monoids, rings, and R-modules.

So I propose the following new definition:

Definition

Let T be an algebraic theory with $M \in Tmod$. Then a T-automorphism $\alpha : M \xrightarrow{\sim} M$ is an **inner automorphism** of M if there is some $[t(x)] \in M^{coh}\langle x \rangle = M^{hom}\langle x \rangle$ such that

$$\alpha = [t(x)]^M : M \to M.$$

Thank you!