

# Isotropy Groups of Quasi-Equational Theories

Jason Parker

(joint with Pieter Hofstra and Philip Scott, University of Ottawa)

Brandon University, Manitoba, Canada

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# Introduction

- **Isotropy** is a (new) mathematical phenomenon with manifestations in category theory, (universal) algebra, and theoretical computer science.
- We will see that isotropy encodes a generalized notion of *conjugation* or *inner automorphism* for many prominent categories in mathematics.

# Motivation

- Recall that an automorphism  $\alpha$  of a group  $G$  is *inner* if there is an element  $s \in G$  such that  $\alpha$  is given by *conjugation* with  $s$ , i.e.

$$(g \in G) \quad \alpha(g) = sgs^{-1}.$$

- It turns out that the inner automorphisms of a group can be characterized *without* mentioning conjugation or group elements at all!

## Motivation

- To see this, observe first that if  $\alpha$  is an inner automorphism of a group  $G$  (induced by  $s \in G$ ), then for each group morphism  $f : G \rightarrow H$  with domain  $G$  we can 'push forward'  $\alpha$  to define an inner automorphism

$$\alpha_f : H \xrightarrow{\sim} H$$

by conjugation with  $f(s) \in H$  (so that  $\alpha_{\text{id}_G} = \alpha$ ), and this family of automorphisms  $(\alpha_f)_f$  is *coherent*, in the sense that it satisfies the following *naturality* property: if  $f : G \rightarrow G'$  and  $f' : G' \rightarrow G''$  are group homomorphisms, then the following diagram commutes:

$$\begin{array}{ccc} G' & \xrightarrow{\alpha_f} & G' \\ f' \downarrow & & \downarrow f' \\ G'' & \xrightarrow{\alpha_{f' \circ f}} & G'' \end{array}$$

## Bergman's Theorem

For a group  $G$ , let us call an *arbitrary* family of automorphisms

$$\left( \alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f) \right)_{\mathbf{dom}(f)=G}$$

with the above naturality property an *extended inner automorphism* of  $G$ . Such a family is a natural automorphism of  $G/\mathbf{Group} \rightarrow \mathbf{Group}$ .

### Theorem (Bergman [1])

Let  $G$  be a group and  $\alpha : G \xrightarrow{\sim} G$  an automorphism of  $G$ . Then  $\alpha$  is an **inner** automorphism of  $G$  iff there is an extended inner automorphism  $(\alpha_f)_f$  of  $G$  with  $\alpha = \alpha_{\mathbf{id}_G}$ .

This provides a completely *element-free* characterization of inner automorphisms of groups! They are exactly those group automorphisms that are 'coherently extendible' along morphisms out of the domain.

# Covariant Isotropy

- We have a functor  $\mathcal{Z} : \mathbf{Group} \rightarrow \mathbf{Group}$  that sends any group  $G$  to its group of extended inner automorphisms  $\mathcal{Z}(G)$ . We refer to  $\mathcal{Z}$  as the *covariant isotropy group (functor)* of the category  $\mathbf{Group}$ .
- In fact, *any* category  $\mathbb{C}$  has a *covariant isotropy group (functor)*

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C} \rightarrow \mathbf{Group}$$

that sends each object  $C \in \mathbb{C}$  to the group of extended inner automorphisms of  $C$ , i.e. families of automorphisms

$$\left( \alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f) \right)_{\mathbf{dom}(f)=C}$$

in  $\mathbb{C}$  with the same naturality property as before, i.e. natural automorphisms of the projection functor  $C/\mathbb{C} \rightarrow \mathbb{C}$ .

# Covariant Isotropy

- We can also turn Bergman's characterization of inner automorphisms in **Group** into a *definition* of inner automorphisms in an arbitrary category  $\mathbb{C}$ : if  $C \in \mathbb{C}$ , we say that an automorphism  $\alpha : C \xrightarrow{\sim} C$  is *inner* if there is an extended inner automorphism  $(\alpha_f)_f \in \mathcal{Z}_{\mathbb{C}}(C)$  with  $\alpha \text{id}_C = \alpha$ .
- Notice that **Group** is the category of (set-based) *models* of an *algebraic theory*, i.e. a set of equational axioms between terms, namely the theory  $\mathbb{T}_{\text{Grp}}$  of groups. So **Group** =  $\mathbb{T}_{\text{Grp}}\mathbf{mod}$ .
- We will generalize ideas from the proof of Bergman's Theorem to give a 'syntactic' characterization of the (extended) inner automorphisms of  $\mathbb{T}\mathbf{mod}$ , i.e. of the covariant isotropy group of  $\mathbb{T}\mathbf{mod}$ , for any so-called *quasi-equational* theory  $\mathbb{T}$ .

# Quasi-Equational Theories

- What is a quasi-equational theory? (Also known as: partial Horn theory, essentially algebraic theory, cartesian theory, finite limit theory.)
- First, we need the notion of a *signature*  $\Sigma$ , which consists of a non-empty set  $\Sigma_{\text{Sort}}$  of *sorts*, and a set  $\Sigma_{\text{Fun}}$  of (typed) *function/operation symbols*.
- For example, the signature for *groups* has one sort  $X$  and three function symbols  $\cdot : X \times X \rightarrow X$ ,  $^{-1} : X \rightarrow X$ , and  $e : X$ . The signature for *categories* has two sorts  $O, A$  and four function symbols **dom**, **cod** :  $A \rightarrow O$ , **id** :  $O \rightarrow A$ , and  $\circ : A \times A \rightarrow A$ .



# Quasi-Equational Theories

- We can then form the set **Term**( $\Sigma$ ) of *terms* over  $\Sigma$ , constructed from variables and function symbols, as well as the set **Horn**( $\Sigma$ ) of *Horn formulas* over  $\Sigma$ , which are finite conjunctions of equations between terms.
- A *quasi-equational theory* over a signature  $\Sigma$  is then a set of *implications* (the *axioms* of  $\mathbb{T}$ ) of the form  $\varphi \Rightarrow \psi$ , with  $\varphi, \psi \in \mathbf{Horn}(\Sigma)$  (see [6]).
- The operation symbols of a quasi-equational theory are only required to be *partially* defined. If  $t$  is a term, we write  $t \downarrow$  as an abbreviation for  $t = t$ , meaning ‘ $t$  is defined’.

## Examples

- Any *algebraic* theory, whose axioms all have the form  $\top \Rightarrow \psi$ , where  $\top$  is the empty conjunction. E.g. the theories of sets, semigroups, (commutative) monoids, (abelian) groups, (commutative) rings with unit, etc. For example, the theory  $\mathbb{T}_{\mathbf{Grp}}$  of groups has the following axioms:

$$\top \Rightarrow x \cdot y \downarrow \wedge x^{-1} \downarrow \wedge e \downarrow,$$

$$\top \Rightarrow x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

$$\top \Rightarrow x \cdot e = x \wedge e \cdot x = x,$$

$$\top \Rightarrow x \cdot x^{-1} = e \wedge x^{-1} \cdot x = e.$$

## Examples

- The theories of categories, groupoids, categories with a terminal object, and cartesian (i.e. finitely complete) categories. E.g. two of the axioms of the theory of categories are

$$g \circ f \downarrow \Rightarrow \mathbf{dom}(g) = \mathbf{cod}(f),$$

$$\mathbf{dom}(g) = \mathbf{cod}(f) \Rightarrow g \circ f \downarrow.$$

- The theory of strict monoidal categories.
- The theory of functors  $\mathcal{J} \rightarrow \mathbb{T}\mathbf{mod}$  for a small category  $\mathcal{J}$  and quasi-equational theory  $\mathbb{T}$ . In particular, the theory of presheaves  $\mathcal{J} \rightarrow \mathbf{Set}$ .

# Proof of Bergman's Theorem

- Let us focus on a specific idea in the proof of Bergman's Theorem.
- Consider the group  $G\langle \mathbf{x} \rangle$  obtained from  $G$  by freely adjoining an indeterminate element  $\mathbf{x}$ . Elements of  $G\langle \mathbf{x} \rangle$  are (reduced) group words in  $\mathbf{x}$  and elements of  $G$ .
- The underlying set of  $G\langle \mathbf{x} \rangle$  can be endowed with a *substitution monoid* structure: given  $w_1, w_2 \in G\langle \mathbf{x} \rangle$ , we set  $w_1 \cdot w_2$  to be the reduction of  $w_1[w_2/\mathbf{x}]$ , and the unit is  $\mathbf{x}$  itself.
- If  $w \in G\langle \mathbf{x} \rangle$ ,  $w$  commutes generically with the group operations if:
  - ▶ In  $G\langle \mathbf{x}_1, \mathbf{x}_2 \rangle$ , the reduction of  $w[\mathbf{x}_1/\mathbf{x}]w[\mathbf{x}_2/\mathbf{x}]$  is  $w[\mathbf{x}_1\mathbf{x}_2/\mathbf{x}]$ ;
  - ▶ In  $G\langle \mathbf{x} \rangle$ , the reduction of  $w^{-1}$  is  $w[\mathbf{x}^{-1}/\mathbf{x}]$ ;
  - ▶ In  $G\langle \mathbf{x} \rangle$ , the reduction of  $w[e/\mathbf{x}]$  in  $G\langle \mathbf{x} \rangle$  is  $e$ .

# Proof of Bergman's Theorem

- E.g. if  $g \in G$ , then the word  $g\mathbf{x}g^{-1} \in G\langle\mathbf{x}\rangle$  commutes generically with the group operations:
  - ▶  $g\mathbf{x}_1g^{-1}g\mathbf{x}_2g^{-1} \sim g\mathbf{x}_1\mathbf{x}_2g^{-1}$
  - ▶  $(g\mathbf{x}g^{-1})^{-1} \sim (g^{-1})^{-1}\mathbf{x}^{-1}g^{-1} \sim g\mathbf{x}^{-1}g^{-1}$ ,
  - ▶  $geg^{-1} \sim gg^{-1} \sim e$ .
- Let  $\mathcal{Z}(G)$  be the group of extended inner automorphisms of  $G$ , and let  $\mathbf{Inv}(G\langle\mathbf{x}\rangle)$  be the subgroup of *invertible* elements of the substitution monoid  $G\langle\mathbf{x}\rangle$ . (E.g.  $g\mathbf{x}g^{-1}$  is invertible, with inverse  $g^{-1}\mathbf{x}g$ .)
- Then the proof of Bergman's Theorem shows that the group  $\mathcal{Z}(G)$  is isomorphic to the subgroup of  $\mathbf{Inv}(G\langle\mathbf{x}\rangle)$  consisting of all words that commute generically with the group operations.

# The Isotropy Group of a Quasi-Equational Theory

- Fix a quasi-equational theory  $\mathbb{T}$  over a signature  $\Sigma$ , and let  $\mathbb{T}\mathbf{mod}$  be the category of (set-based) models of  $\mathbb{T}$ . For simplicity, we will generally assume (in this talk) that  $\mathbb{T}$  is single-sorted.
- We will now give a *logical/syntactic* characterization of the covariant isotropy group

$$\mathcal{Z}_{\mathbb{T}} : \mathbb{T}\mathbf{mod} \rightarrow \mathbf{Group}$$

of  $\mathbb{T}\mathbf{mod}$ .

- Fix  $M \in \mathbb{T}\mathbf{mod}$ . As for groups, we can construct a  $\mathbb{T}$ -model  $M\langle \mathbf{x} \rangle$ , which is the coproduct of  $M$  with the free  $\mathbb{T}$ -model on one generator  $\mathbf{x}$ . Elements of  $M\langle \mathbf{x} \rangle$  are (equivalence classes of)  $\Sigma$ -terms over  $\mathbf{x}$  and elements of  $M$ . We can then endow the underlying set of  $M\langle \mathbf{x} \rangle$  with a *substitution monoid* structure, in the same way as for groups.

# The Isotropy Group of a Quasi-Equational Theory

In my thesis, I proved:

## Theorem ([7])

Let  $\mathbb{T}$  be a quasi-equational theory over a (single-sorted) signature  $\Sigma$ . For any  $M \in \mathbb{T}\mathbf{mod}$ , the covariant isotropy group  $\mathcal{Z}_{\mathbb{T}}(M)$ , i.e. the group of extended inner automorphisms of  $M$ , is isomorphic to the group of **invertible** elements  $t$  of the substitution monoid  $M\langle \mathbf{x} \rangle$  that **commute generically with** the function symbols of  $\Sigma$ , in the sense that if  $f$  is any  $n$ -ary function symbol of  $\Sigma$ , then

$$t[f(\mathbf{x}_1, \dots, \mathbf{x}_n)/\mathbf{x}] = f(t[\mathbf{x}_1/\mathbf{x}], \dots, t[\mathbf{x}_n/\mathbf{x}])$$

holds in  $M\langle \mathbf{x}_1, \dots, \mathbf{x}_n \rangle$  (the coproduct of  $M$  with the free  $\mathbb{T}$ -model on  $n$  generators  $\mathbf{x}_1, \dots, \mathbf{x}_n$ ).

# The Isotropy Group of a Quasi-Equational Theory

- In particular, an automorphism  $\alpha : M \xrightarrow{\sim} M$  in  $\mathbb{T}\mathbf{mod}$  is *inner* iff there is some  $t \in \mathcal{Z}_{\mathbb{T}}(M)$  that *induces*  $\alpha$ , i.e.

$$(m \in M) \quad \alpha(m) = t[m/\mathbf{x}] \in M.$$

- Thus, Bergman's (syntactic) characterization of the (extended) inner automorphisms of  $\mathbf{Group} = \mathbb{T}_{\mathbf{Grp}}\mathbf{mod}$  extends to the category  $\mathbb{T}\mathbf{mod}$  of (set-based) models of *any* quasi-equational theory  $\mathbb{T}$ .



## Examples

- If  $\mathbb{T}$  is the theory of sets, then  $\mathbb{T}$  has trivial isotropy group, i.e.  $\mathcal{Z}_{\mathbb{T}}(S) \cong \{\mathbf{x}\}$  for any set  $S$ , so the only inner automorphism of a set is the *identity* function.
- If  $\mathbb{T}$  is the theory of groups, then Bergman proved  $\forall G \in \mathbb{T}\mathbf{mod} = \mathbf{Group}$  that

$$\mathcal{Z}_{\mathbb{T}}(G) \cong \{g\mathbf{x}g^{-1} \in G\langle\mathbf{x}\rangle \mid g \in G\} \cong G.$$

- If  $\mathbb{T}$  is the theory of monoids, then  $\forall M \in \mathbb{T}\mathbf{mod} = \mathbf{Mon}$  we have

$$\mathcal{Z}_{\mathbb{T}}(M) \cong \{m\mathbf{x}m^{-1} \in M\langle\mathbf{x}\rangle \mid m \text{ is invertible in } M\} \cong \mathbf{Inv}(M).$$

## Examples

- If  $\mathbb{T}$  is the theory of abelian groups, then  $\forall G \in \mathbb{T}\mathbf{mod} = \mathbf{Ab}$  we have

$$\mathcal{Z}_{\mathbb{T}}(G) \cong \{\mathbf{x}, -\mathbf{x}\} \cong \mathbb{Z}_2.$$

- If  $\mathbb{T}$  is the theory of commutative monoids or unital rings, then the isotropy group of  $\mathbb{T}$  is trivial.
- If  $\mathbb{T}$  is the theory of (not necessarily commutative) unital rings, then  $\forall R \in \mathbb{T}\mathbf{mod} = \mathbf{Ring}$  we have

$$\mathcal{Z}_{\mathbb{T}}(R) \cong \{rxr^{-1} \in R\langle \mathbf{x} \rangle \mid r \in R \text{ is a unit}\} \cong \mathbf{Unit}(R).$$

- If  $\mathbb{T}$  is the theory of categories, groupoids, or categories with a terminal object, then the isotropy group of  $\mathbb{T}$  is trivial.

## Examples

- If  $\mathbb{T}$  is the theory of strict monoidal categories, then for any strict monoidal category  $\mathbb{C}$  we have

$$\mathcal{Z}_{\mathbb{T}}(\mathbb{C}) \cong \mathbf{Inv} \left( \mathbb{C}_O, \otimes^{\mathbb{C}}, e^{\mathbb{C}} \right),$$

the group of invertible elements of the object monoid  $(\mathbb{C}_O, \otimes^{\mathbb{C}}, e^{\mathbb{C}})$  of  $\mathbb{C}$ . In particular, if  $F : \mathbb{C} \xrightarrow{\sim} \mathbb{C}$  is a (strict monoidal) automorphism of a strict monoidal category  $\mathbb{C}$ , then  $F$  is *inner* iff there is some invertible object  $c \in \mathbb{C}$  such that  $F$  is given by *conjugation* with  $c$ , i.e.

$$(a \in \mathbb{C}_O) \quad F(a) = c \otimes a \otimes c^{-1}$$

and

$$(f \in \mathbb{C}_A) \quad F(f) = \mathbf{id}_c \otimes f \otimes \mathbf{id}_{c^{-1}}.$$

# Isotropy Groups of Functor Categories

- We can also characterize the covariant isotropy groups of *functor categories* of the form  $\mathbb{T}\mathbf{mod}^{\mathcal{J}}$ , for a quasi-equational theory  $\mathbb{T}$  and small category  $\mathcal{J}$ . In particular, we can characterize the covariant isotropy groups of presheaf categories  $\mathbf{Set}^{\mathcal{J}}$ .
- Fix a quasi-equational theory  $\mathbb{T}$ . Given a small category  $\mathcal{J}$ , we can define a quasi-equational theory  $\mathbb{T}^{\mathcal{J}}$  whose models are functors  $\mathcal{J} \rightarrow \mathbb{T}\mathbf{mod}$ , i.e.

$$\mathbb{T}^{\mathcal{J}}\mathbf{mod} \cong \mathbb{T}\mathbf{mod}^{\mathcal{J}}.$$

# Isotropy Groups of Functor Categories

In my thesis, I then proved the following theorem:

## Theorem ([7])

Let  $\mathbb{T}$  be a (single-sorted) quasi-equational theory (satisfying a few technical assumptions), and let  $\mathcal{J}$  be a small category, with  $\mathbf{Aut}(\mathbf{Id}_{\mathcal{J}})$  the group of natural automorphisms of  $\mathbf{Id}_{\mathcal{J}} : \mathcal{J} \rightarrow \mathcal{J}$  (which we may call the **global isotropy group** of  $\mathcal{J}$ ). For any functor  $F : \mathcal{J} \rightarrow \mathbb{T}\mathbf{mod}$ , we have

$$\mathcal{Z}_{\mathbb{T}\mathbf{mod}^{\mathcal{J}}}(F) \cong \mathbf{lim}(\mathcal{Z}_{\mathbb{T}} \circ F) \times \mathbf{Aut}(\mathbf{Id}_{\mathcal{J}}) \in \mathbf{Group}.$$

In particular, for any functor  $F : \mathcal{J} \rightarrow \mathbf{Set}$ , we have

$$\mathcal{Z}_{\mathbf{Set}^{\mathcal{J}}}(F) \cong \mathbf{Aut}(\mathbf{Id}_{\mathcal{J}}).$$

# Isotropy Groups of Functor Categories

- In particular, if  $F : \mathcal{J} \rightarrow \mathbf{Set}$  is a functor and  $\alpha : F \xrightarrow{\sim} F$  is an automorphism, then  $\alpha$  is *inner* iff there is some  $\psi \in \mathbf{Aut}(\mathbf{Id}_{\mathcal{J}})$  with

$$(k \in \mathcal{J}) \quad \alpha_k = F(\psi_k) : F(k) \xrightarrow{\sim} F(k).$$

- So the covariant isotropy group functor  $\mathcal{Z} : \mathbf{Set}^{\mathcal{J}} \rightarrow \mathbf{Group}$  is *constant* on the global isotropy group  $\mathbf{Aut}(\mathbf{Id}_{\mathcal{J}})$  of  $\mathcal{J}$ .

# Isotropy Groups of $G$ -Sets

- For any group  $G$ , the covariant isotropy group functor  $\mathcal{Z} : \mathbf{Set}^G \rightarrow \mathbf{Group}$  of the category of  $G$ -sets is *constant* on the centre  $Z(G)$  of the group  $G$ .
- More generally, for any monoid  $M$ , the covariant isotropy group functor  $\mathcal{Z} : \mathbf{Set}^M \rightarrow \mathbf{Group}$  of the category of  $M$ -sets is *constant* on the group  $\mathbf{Inv}(Z(M))$  of invertible elements of the centre of  $M$ .

## Connections with Topos Theory

- If  $\mathbb{T}$  is a quasi-equational theory, then  $\mathbb{T}$  has a *classifying topos*  $\mathcal{B}(\mathbb{T})$ , which is a cocomplete topos that has a *universal model* of  $\mathbb{T}$  and classifies all topos-theoretic models of  $\mathbb{T}$  ([4], [5]).
- It has been shown that any Grothendieck topos  $\mathcal{E}$  has a canonical internal group object called the *isotropy group* of the topos, which acts canonically on every object of the topos and formally generalizes the notion of conjugation ([3]).
- The covariant isotropy group  $\mathcal{Z}_{\mathbb{T}}$  of a quasi-equational theory  $\mathbb{T}$  is in fact the isotropy group object of the classifying topos  $\mathcal{B}(\mathbb{T})$  of  $\mathbb{T}$  ([3], [4]).



## Conclusions







- Bergman's *element-free* characterization of the inner automorphisms of groups can be used to *define* inner automorphisms in arbitrary categories.
- We have extended Bergman's *syntactic* characterization of the (extended) inner automorphisms of groups, i.e. of the covariant isotropy group of  $\mathbf{Group} = \mathbb{T}_{\mathbf{Grp}}\mathbf{mod}$ , to the covariant isotropy group of  $\mathbb{T}\mathbf{mod}$  for *any* quasi-equational theory  $\mathbb{T}$ .
- Using this characterization, we have obtained concrete descriptions of the (extended) inner automorphisms in several different categories:  $\mathbf{Set}$ ,  $\mathbf{Group}$ ,  $\mathbf{Mon}$ ,  $\mathbf{Ab}$ ,  $\mathbf{Ring}$ ,  $\mathbf{Cat}$ ,  $\mathbf{StrMonCat}$ ,  $\mathbb{T}\mathbf{mod}^{\mathcal{J}}$ ,  $\mathbf{Set}^{\mathcal{J}}$ ,  $\dots$
- This work also represents a contribution to the more general project of characterizing the isotropy group objects of Grothendieck toposes.

## Some Future Directions

- Given (disjoint) theories  $\mathbb{T}_1$  and  $\mathbb{T}_2$ , characterize the covariant isotropy group of the category of models of  $\mathbb{T}_1$  in  $\mathbb{T}_2\text{mod}$  (i.e. the category of models of  $\mathbb{T}_1 \otimes \mathbb{T}_2$ ) in terms of the covariant isotropy groups of  $\mathbb{T}_1$  and  $\mathbb{T}_2$  (subsuming the examples of strict monoidal categories and functor categories  $\mathbb{T}\mathbf{mod}^{\mathcal{J}}$ ).
- Characterize the covariant isotropy groups of Grothendieck toposes, i.e. categories  $\text{Sh}(\mathbb{C}, J)$  in terms of the (small) site presentation  $(\mathbb{C}, J)$ . Categories of the form  $\text{Sh}(\mathbb{C}, J)$  are categories of models for an (infinitary) quasi-equational theory.
- Characterize the *contravariant* isotropy groups of quasi-equational theories.

Thank you!

# References I

-  G. Bergman. An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange. *Publicacions Matematiques* 56, 91-126, 2012.
-  P. Freyd. Core algebra revisited. *Theoretical Computer Science* 374, 193-200, 2007.
-  J. Funk, P. Hofstra, B. Steinberg. Isotropy and crossed toposes. *Theory and Applications of Categories* 26, 660-709, 2012.
-  P. T. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Clarendon Press, 2002.
-  S. Mac Lane, I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Springer-Verlag, 1992.
-  E. Palmgren, S.J. Vickers. Partial Horn logic and cartesian categories. *Annals of Pure and Applied Logic* 145, 314-353, 2007.

## References II



J. Parker. Isotropy Groups of Quasi-Equational Theories. PhD thesis, University of Ottawa, 2020.