Enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities

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Motivation

- Several structure-semantics adjunctions and monad-theory equivalences have been established in category theory.

- In [7], Lawvere established a structure-semantics adjunction between Lawvere theories and *tractable* **Set**-valued functors, which was later generalized by Linton [8]. For a complete and well-powered closed category $\mathscr{V}$, Dubuc [4] proved a structure-semantics adjunction between *$\mathscr{V}$-theories* and *tractable* $\mathscr{V}$-valued $\mathscr{V}$-functors.

Motivation

- Linton [8] also showed that there is an equivalence between Lawvere theories and finitary monads on **Set**. Lucyshyn-Wright [10] generalized this by showing that if $\mathscr{J} \hookrightarrow \mathscr{V}$ is any *eleutheric system* of arities in a closed category $\mathscr{V}$, then there is an equivalence between $\mathscr{J}$-theories and $\mathscr{J}$*-ary* $\mathscr{V}$-monads on $\mathscr{V}$.

- Building on earlier work of Power and Nishizawa [14, 13], Bourke and Garner [2] recently showed that if $\mathscr{J} \hookrightarrow \mathscr{C}$ is any small subcategory of arities in a locally presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally presentable closed category $\mathscr{V}$, then there is an equivalence between $\mathscr{J}$*-theories* and $\mathscr{J}$*-nervous* $\mathscr{V}$-monads on $\mathscr{C}$.

- Neither equivalence subsumes the other; can both equivalences, along with the aforementioned structure-semantics adjunctions, be captured by a common framework that also yields new examples?

Objectives

- That is the subject of this talk: we have developed a general framework for studying enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities, which specializes to recover the aforementioned results and also yields new examples.
- More specifically, given a subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ over a closed category $\mathscr{V}$, we will identify hypotheses on these data that entail a structure-semantics adjunction, a monad-theory equivalence, a rich theory of *presentations* for monads and theories, and more.

Basic definitions

- We fix a **subcategory of arities** $j : \mathscr{J} \hookrightarrow \mathscr{C}$, i.e. a full and dense sub-$\mathscr{V}$-category, in a $\mathscr{V}$-category $\mathscr{C}$ over a symmetric monoidal closed category $\mathscr{V}$. Since we do not assume that $\mathscr{J}$ is small or that $\mathscr{V}$ is (co)complete, we also fix a suitable universe extension $\mathscr{V} \hookrightarrow \mathscr{V}'$.

- We have a fully faithful $\mathscr{V}'$-functor

$$N_j : \mathscr{C} \to [\mathscr{J}^{\mathbf{op}}, \mathscr{V}]$$

$$N_j C = \mathscr{C}(j-, C)$$

that we call the j-**nerve** $\mathscr{V}'$-functor. The presheaves in its essential image are called j-**nerves**.

Pretheories and their algebras

- (Linton [8], Diers [3], Bourke-Garner [2]) A $\mathscr{J}$-**pretheory** is just an identity-on-objects $\mathscr{V}$-functor $\tau : \mathscr{J}^{\mathbf{op}} \to \mathscr{T}$, while a $\mathscr{J}$-**theory** is a $\mathscr{J}$-pretheory $\mathscr{T}$ such that each $\mathscr{T}(J, \tau-) : \mathscr{J}^{\mathbf{op}} \to \mathscr{V}$ ($J \in \mathbf{ob}\,\mathscr{J}$) is a j-nerve. We have the category $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-pretheories and its full subcategory $\mathbf{Th}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-theories.

- Let $\mathscr{T}$ be a $\mathscr{J}$-pretheory. The $\mathscr{V}'$-category $\mathscr{T}$-Alg of **(concrete)** $\mathscr{T}$-**algebras** is defined by the following pullback in $\mathscr{V}'$-CAT:

$$\begin{array}{ccc} \mathscr{T}\text{-Alg} & \longrightarrow & [\mathscr{T}, \mathscr{V}] \\ {\scriptstyle U^{\mathscr{T}}}\big\downarrow & & \big\downarrow{\scriptstyle [\tau,1]} \\ \mathscr{C} & \xrightarrow[N_j]{} & [\mathscr{J}^{\mathbf{op}}, \mathscr{V}]. \end{array}$$

Amenable subcategories of arities

- A $\mathscr{J}$-pretheory $\mathscr{T}$ is **admissible** if the $\mathscr{V}'$-category $\mathscr{T}$-Alg is actually a $\mathscr{V}$-category, and $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}$ has a left adjoint.
- The subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **amenable** if every $\mathscr{J}$-theory is admissible, and is **strongly amenable** if every $\mathscr{J}$-pretheory $\mathscr{T}$ is admissible.

$\mathscr{J}$-tractable $\mathscr{V}$-categories

- A **$\mathscr{J}$-tractable $\mathscr{V}$-category over** $\mathscr{C}$ is a $\mathscr{V}$-category $G : \mathscr{A} \to \mathscr{C}$ over $\mathscr{C}$ such that $\mathscr{C}$ admits the weighted limit $\{\mathscr{C}(J, G-), G\}$ for each $J \in \mathbf{ob}\,\mathscr{J}$. Then $\mathscr{J}$-$\mathbf{Tract}(\mathscr{C})$ is the full subcategory of $\mathscr{V}$-CAT$/\mathscr{C}$ consisting of the $\mathscr{J}$-tractable $\mathscr{V}$-categories over $\mathscr{C}$.

- Let $\mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ be the full subcategory of $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ consisting of the *admissible* $\mathscr{J}$-pretheories. We define a **semantics** functor

$$\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$$

by

$$\mathbf{Sem}\,\mathscr{T} = \left(U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}\right)$$

for each admissible $\mathscr{J}$-pretheory $\mathscr{T}$.

$\mathscr{J}$-structure

- Let $G : \mathscr{A} \to \mathscr{C}$ be a $\mathscr{J}$-tractable $\mathscr{V}$-category over $\mathscr{C}$. We define a $\mathscr{J}$-theory $\tau_G : \mathscr{J}^{\mathbf{op}} \to \mathbf{Str}\,G$, the **$\mathscr{J}$-structure of G**, by taking the (identity-on-objects, fully faithful) factorization of the composite $\mathscr{V}'$-functor

$$\begin{array}{ccccc} \mathscr{J}^{\mathbf{op}} & \xrightarrow{\;j^{\mathbf{op}}\;} & \mathscr{C}^{\mathbf{op}} & \xrightarrow{\;N_{G^{\mathbf{op}}}\;} & [\mathscr{A}, \mathscr{V}]. \\ & {\scriptstyle \tau_G}\searrow & & \nearrow & \\ & & \mathbf{Str}\,G & & \end{array}$$

(Since G is $\mathscr{J}$-tractable, $\mathbf{Str}\,G$ is indeed a $\mathscr{V}$-category and moreover a $\mathscr{J}$-theory).

The structure-semantics adjunction

A $\mathscr{J}$**-algebraic** $\mathscr{V}$**-category over** $\mathscr{C}$ is a $\mathscr{V}$-category over $\mathscr{C}$ in the essential image of **Sem**; we let $\mathscr{J}$-$\mathbf{Alg}(\mathscr{C})$ be the full subcategory of $\mathscr{J}$-$\mathbf{Tract}(\mathscr{C})$ consisting of these objects.

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. Then the semantics functor $\mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$ *has a left adjoint* **Str** *that sends each $\mathscr{J}$-tractable $\mathscr{V}$-category over $\mathscr{C}$ to its $\mathscr{J}$-structure. This adjunction is idempotent, and restricts to an adjoint equivalence*

$$\mathbf{Th}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \underset{\mathbf{Str}}{\overset{\mathbf{Sem}}{\rightleftarrows}} \mathscr{J}\text{-}\mathbf{Alg}(\mathscr{C})$$

between $\mathscr{J}$-theories and $\mathscr{J}$-algebraic $\mathscr{V}$-categories over $\mathscr{C}$.

The monad-pretheory adjunction

- Given an admissible $\mathscr{J}$-pretheory $\mathscr{T}$, the $\mathscr{V}$-functor $U^{\mathscr{T}} : \mathscr{T}\text{-Alg} \to \mathscr{C}$ is strictly monadic, and hence **Sem** corestricts to the full subcategory $\mathbf{Monadic}^!(\mathscr{C}) \hookrightarrow \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$ of strictly monadic $\mathscr{V}$-categories over $\mathscr{C}$.

- Let $\mathscr{J}$ be amenable. Then because $\mathbf{Monadic}^!(\mathscr{C}) \simeq \mathbf{Mnd}(\mathscr{C})^{\mathbf{op}}$, the structure-semantics adjunction yields an idempotent adjunction

$$\mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Mnd}(\mathscr{C}), \quad \Psi \dashv \Phi$$

where Φ sends a $\mathscr{V}$-monad $\mathbb{T}$ to its **Kleisli** $\mathscr{J}$**-theory**, while Ψ sends an admissible $\mathscr{J}$-pretheory $\mathscr{T}$ to the free $\mathscr{T}$-algebra $\mathscr{V}$-monad on $\mathscr{C}$.

The monad-theory equivalence

A $\mathscr{V}$-monad $\mathbb{T}$ on $\mathscr{C}$ is $\mathscr{J}$-**nervous** if $\mathbb{T} \cong \Psi\mathscr{T}$ for some admissible $\mathscr{J}$-pretheory $\mathscr{T}$ (there is also a more technical definition that does not involve pretheories).

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an amenable subcategory of arities. Then the idempotent monad-pretheory adjunction $\Psi \dashv \Phi$ restricts to an adjoint equivalence

$$\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \underset{\Phi}{\overset{\Psi}{\rightleftarrows}} \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$$

between $\mathscr{J}$-theories and $\mathscr{J}$-nervous $\mathscr{V}$-monads, which commutes with semantics in an appropriate sense. Also $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ is reflective, while $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Mnd}(\mathscr{C})$ is coreflective.

Additional consequences of strong amenability

We now suppose that $\mathscr{V}$ is complete and cocomplete, that $\mathscr{C}$ is cocomplete and cotensored, and that $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is small and strongly amenable.

Proposition

$\mathbf{Preth}_{\mathscr{J}}(\mathscr{C}), \mathbf{Th}_{\mathscr{J}}(\mathscr{C})$, *and* $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ *are all cocomplete, and small colimits therein are sent to limits in* $\mathscr{V}\text{-CAT}/\mathscr{C}$ *by the respective semantics functors.*

Monadicity over signatures

A $\mathscr{J}$-**signature** is a $\mathscr{V}$-functor $\Sigma : \mathbf{ob}\,\mathscr{J} \to \mathscr{C}$, i.e. an $\mathbf{ob}\,\mathscr{J}$-indexed family of objects of $\mathscr{C}$. We have a category $\mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ of $\mathscr{J}$-signatures, and a forgetful functor $\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ defined by

$$\mathcal{U}\mathbb{T} = (TJ)_{J \in \mathscr{J}}\,.$$

Theorem

The functor $\mathcal{U} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}(\mathscr{C})$ is monadic, and hence every $\mathscr{J}$-nervous $\mathscr{V}$-monad has a $\mathscr{J}$-presentation. Moreover, every $\mathscr{J}$-presentation P presents a $\mathscr{J}$-nervous $\mathscr{V}$-monad $\mathbb{T}_P$ with $\mathbb{T}_P\text{-Alg} \cong P\text{-Alg}$ in $\mathscr{V}\text{-CAT}/\mathscr{C}$.

Some other nice consequences

We now also suppose that $\mathscr{T}$-Alg has conical coequalizers of reflexive pairs for each $\mathscr{J}$-pretheory $\mathscr{T}$.

Theorem

Let $H : \mathscr{T} \to \mathscr{U}$ be a morphism of $\mathscr{J}$-pretheories. Then the algebraic $\mathscr{V}$-functor $H^* = \mathbf{Sem}\ H : \mathscr{U}\text{-Alg} \to \mathscr{T}\text{-Alg}$ is strictly monadic.

Theorem

Let $\mathscr{T}$ be a $\mathscr{J}$-pretheory. Then the full sub-$\mathscr{V}$-category $\mathscr{T}\text{-Alg} \hookrightarrow [\mathscr{T}, \mathscr{V}]$ is reflective.

Summary so far...

- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is amenable, then we have a structure-semantics adjunction $\mathbf{Str} \dashv \mathbf{Sem} : \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$; a monad-theory equivalence $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$; and the reflectivity of $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Preth}^{\mathsf{a}}_{\mathscr{J}}(\mathscr{C})$ and coreflectivity of $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \hookrightarrow \mathbf{Mnd}(\mathscr{C})$.
- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is small and strongly amenable and $\mathscr{C}, \mathscr{V}$ are sufficiently (co)complete, then we also have a monad-pretheory adjunction $\Psi \dashv \Phi : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$; the (algebraic) cocompleteness of $\mathbf{Preth}_{\mathscr{J}}(\mathscr{C}), \mathbf{Th}_{\mathscr{J}}(\mathscr{C}), \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$; a rich theory of presentations for $\mathscr{J}$-nervous $\mathscr{V}$-monads (and hence $\mathscr{J}$-theories); the strict monadicity of algebraic $\mathscr{V}$-functors; and the reflectivity of $\mathscr{T}$-algebras in presheaves (and more...).

First example: eleutheric subcategories of arities

- A subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **eleutheric** [10, 12] if every $\mathscr{V}$-functor $H : \mathscr{J} \to \mathscr{C}$ has a left Kan extension along j that is preserved by each $\mathscr{C}(J, -) : \mathscr{C} \to \mathscr{V}$ ($J \in \mathbf{ob}\,\mathscr{J}$). For example:
  - The full sub-$\mathscr{V}$-category of enriched α-presentable objects in a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally α-presentable $\mathscr{V}$.
  - The "strongly finitary" subcategory of arities $j : \mathbf{SF}(\mathscr{V}) \hookrightarrow \mathscr{V}$ consisting of the finite copowers of the terminal object in a complete and cocomplete cartesian closed $\mathscr{V}$.
  - Just the unit object $\{I\} \hookrightarrow \mathscr{V}$ in any closed category $\mathscr{V}$.
  - The "unrestricted" subcategory of arities $1_{\mathscr{C}} : \mathscr{C} \hookrightarrow \mathscr{C}$ in any $\mathscr{V}$-category $\mathscr{C}$.
  - The Yoneda embedding $\mathbf{y} : \mathscr{A}^{\mathbf{op}} \hookrightarrow [\mathscr{A}, \mathscr{V}]$ for any small $\mathscr{V}$-category $\mathscr{A}$.
  - Any free Ψ-cocompletion $j : \mathscr{J} \hookrightarrow \mathscr{C}$ of a small $\mathscr{V}$-category $\mathscr{J}$ under a class of small weights Ψ.

First example: eleutheric subcategories of arities

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be an eleutheric subcategory of arities. Then $\mathscr{J}$ is amenable.

- We will observe below that most of the above examples satisfy an additional *boundedness* property that also makes them **strongly** amenable.
- If $j = 1_{\mathscr{V}} : \mathscr{V} \to \mathscr{V}$, then we recover Dubuc's structure-semantics adjunction [4] between $\mathscr{V}$-theories and tractable $\mathscr{V}$-valued $\mathscr{V}$-functors, and his equivalence between $\mathscr{V}$-theories and arbitrary $\mathscr{V}$-monads on $\mathscr{V}$.
- If $\mathscr{C} = \mathscr{V}$ and $j : \mathscr{J} \hookrightarrow \mathscr{V}$ is an eleutheric **system** of arities (i.e. contains I and is closed under $\otimes$), then we recover Lucyshyn-Wright's equivalence [10] between $\mathscr{J}$-theories and $\mathscr{J}$*-ary* $\mathscr{V}$-monads on $\mathscr{V}$.

Second example: bounded subcategories of arities

For this example, we make the following background assumptions:

- $\mathscr{V}$ is complete and cocomplete and has an enriched factorization system $(\mathscr{E}, \mathscr{M})$ [9].
- $\mathscr{C}$ is cocomplete and cotensored and has a *compatible* enriched factorization system $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ [12], and $\mathscr{C}$ has arbitrary $\mathscr{E}_{\mathscr{C}}$-cointersections; moreover, $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ is proper or $\mathscr{C}$ is $\mathscr{E}_{\mathscr{C}}$-cowellpowered.

A small subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is **bounded** if each $J \in \mathbf{ob}\, \mathscr{J}$ is bounded (in the sense of [12]). If $\mathscr{C}$ is a locally bounded $\mathscr{V}$-category [11] over a locally bounded closed category $\mathscr{V}$, then any small $\mathscr{J} \hookrightarrow \mathscr{C}$ is automatically bounded.

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be a (small) subcategory of arities that is contained in some bounded and eleutheric subcategory of arities. Then $\mathscr{J}$ is strongly amenable, and $\mathscr{T}$-Alg has small conical colimits for every $\mathscr{J}$-pretheory $\mathscr{T}$.

Second example: bounded subcategories of arities

- For example: most of the above examples of eleutheric subcategories of arities are also bounded, and hence strongly amenable. Also, any small subcategory of arities in a locally presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally presentable $\mathscr{V}$ is contained in a bounded and eleutheric subcategory of arities, from which we recover the monad-pretheory adjunction and monad-theory equivalence of Bourke and Garner [2].
- By dropping the requirement of eleuthericity and strengthening the notion of boundedness in certain ways, we can also obtain further examples of strongly amenable subcategories of arities.

Locally bounded examples

A $\mathscr{V}$-category $\mathscr{C}$ is $\mathscr{V}$-**sketchable** if $\mathscr{C}$ is equivalent to the $\mathscr{V}$-category Φ-$\mathbf{Cts}(\mathcal{T}, \mathscr{V})$ of models of a small Φ-theory $\mathcal{T}$ for a class of small weights Φ.

Theorem

*Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be any small subcategory of arities in a $\mathscr{V}$-sketchable $\mathscr{V}$-category $\mathscr{C}$ over a locally bounded closed category $\mathscr{V}$. Then $\mathscr{J}$ is strongly amenable. If $\mathscr{V}$ is $\mathscr{E}$-cowellpowered, then $\mathscr{T}$-*Alg *is locally bounded (and hence cocomplete) for any $\mathscr{J}$-pretheory $\mathscr{T}$, and* $\mathbb{T}$-Alg *is locally bounded for any $\mathscr{J}$-nervous $\mathscr{V}$-monad $\mathbb{T}$.*

This provides a second method for recovering the main results of Bourke-Garner [2], because every locally presentable $\mathscr{V}$ is locally bounded and every locally presentable $\mathscr{V}$-category $\mathscr{C}$ is $\mathscr{V}$-sketchable [6].

Locally bounded examples

Since $\mathscr{V}$ itself is $\mathscr{V}$-sketchable, we may take $\mathscr{C} = \mathscr{V}$ and obtain the following:

Theorem

Let $j : \mathscr{J} \hookrightarrow \mathscr{V}$ be any small subcategory of arities in a locally bounded closed category $\mathscr{V}$. Then $\mathscr{J}$ is strongly amenable, and $\mathscr{T}$-Alg is cocomplete for each $\mathscr{J}$-pretheory $\mathscr{T}$.

As shown in [11], we have the following examples of locally bounded closed categories: any locally presentable closed category; any cocomplete locally cartesian closed category with a small generator (e.g. Dubuc's concrete quasitoposes [5] and the convenient categories of smooth spaces of [1]); any topological category over **Set** with its canonical ("separate continuity") symmetric monoidal closed structure (e.g. **Top** and **Meas**); and many convenient (cartesian closed) categories of topological spaces.

Locally bounded examples

E.g. let $\mathscr{V}$ be any "convenient" cartesian closed category of topological spaces, and let $\mathscr{J} = \mathsf{Fin}_{\mathscr{V}} \hookrightarrow \mathscr{V}$ be (a skeleton of) the small full sub-$\mathscr{V}$-category consisting of the finite spaces. Then $\mathsf{Fin}_{\mathscr{V}}$ is strongly amenable, and hence we obtain an equivalence

$$\mathbf{Th}_{\mathsf{Fin}_{\mathscr{V}}}(\mathscr{V}) \simeq \mathbf{Mnd}_{\mathsf{Fin}_{\mathscr{V}}}(\mathscr{V})$$

between $\mathsf{Fin}_{\mathscr{V}}$-theories and $\mathsf{Fin}_{\mathscr{V}}$-nervous $\mathscr{V}$-monads on $\mathscr{V}$, along with all the other nice results for strongly amenable subcategories of arities.

In summary...

- We have developed a general framework for enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities. If $\mathscr{J}$ is amenable (every $\mathscr{J}$-theory has free algebras), then we have a structure-semantics adjunction

$$\mathbf{Str} \dashv \mathbf{Sem} : \mathbf{Preth}^{\mathrm{a}}_{\mathscr{J}}(\mathscr{C})^{\mathbf{op}} \to \mathscr{J}\text{-}\mathbf{Tract}(\mathscr{C})$$

  and a monad-theory equivalence $\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$.

- If $\mathscr{C}, \mathscr{V}$ are sufficiently (co)complete and $\mathscr{J}$ is small and strongly amenable (every $\mathscr{J}$-pretheory has free algebras), then we also have a monad-pretheory adjunction $\Psi \dashv \Phi : \mathbf{Mnd}(\mathscr{C}) \to \mathbf{Preth}_{\mathscr{J}}(\mathscr{C})$ and a rich theory of presentations and algebraic colimits for $\mathscr{J}$-theories and $\mathscr{J}$-nervous $\mathscr{V}$-monads.

In summary...

- Many previously studied subcategories of arities are (strongly) amenable, from which we obtain many of the enriched structure-semantics adjunctions and monad-theory equivalences already established in the literature.

- Every small subcategory of arities in a $\mathscr{V}$-sketchable $\mathscr{V}$-category $\mathscr{C}$ over a locally bounded closed category $\mathscr{V}$ is strongly amenable; in particular, we may take $\mathscr{C} = \mathscr{V}$ itself. Examples of such $\mathscr{V}$ include many convenient categories of spaces.

Thank you!

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References I

[1] John C. Baez and Alexander E. Hoffnung, *Convenient categories of smooth spaces*, Trans. Amer. Math. Soc. **363** (2011), no. 11, 5789–5825.

[2] John Bourke and Richard Garner, *Monads and theories*, Adv. Math. **351** (2019), 1024–1071.

[3] Yves Diers, *Foncteur pleinement fidéle dense classant les algébres*, Cahiers Topologie Géom. Différentielle Catég. **17** (1976), no. 2, 171–186.

[4] Eduardo J. Dubuc, *Enriched semantics-structure (meta) adjointness*, Rev. Un. Mat. Argentina **25** (1970/71), 5–26.

[5] ———, *Concrete quasitopoi*, Applications of sheaves, Lecture Notes in Math., vol. 753, Springer, Berlin, 1979, pp. 239–254.

References II

[6] G. M. Kelly, *Structures defined by finite limits in the enriched context I*, Cahiers Topologie Géom. Différentielle Catég. **23** (1982), no. 1, 3–42.

[7] F. William Lawvere, *Functorial semantics of algebraic theories and some algebraic problems in the context of functorial semantics of algebraic theories*, Repr. Theory Appl. Categ. (2004), no. 5, 1–121.

[8] F. E. J. Linton, *An outline of functorial semantics*, Sem. on Triples and Categorical Homology Theory (ETH, Zürich, 1966/67), Springer, Berlin, 1969, pp. 7–52.

[9] Rory B. B. Lucyshyn-Wright, *Enriched factorization systems*, Theory Appl. Categ. **29** (2014), No. 18, 475–495.

[10] ______, *Enriched algebraic theories and monads for a system of arities*, Theory Appl. Categ. **31** (2016), No. 5, 101–137.

References III

[11] Rory B.B. Lucyshyn-Wright and Jason Parker, *Locally bounded enriched categories*, Preprint, arXiv:2110.07072, 2021.

[12] ______, *Presentations and algebraic colimits of enriched monads for a subcategory of arities*, Preprint, arXiv:2201.03466, 2022.

[13] Koki Nishizawa and John Power, *Lawvere theories enriched over a general base*, J. Pure Appl. Algebra **213** (2009), no. 3, 377–386.

[14] John Power, *Enriched Lawvere theories*, Theory Appl. Categ. **6** (1999), 83–93.