Locally bounded enriched categories

Jason Parker
(j.w.w. Rory Lucyshyn-Wright)

Brandon University, Manitoba

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Introduction

- Locally bounded ordinary categories were (implicitly) introduced by Freyd and Kelly in
  - [3] P.J. Freyd and G.M. Kelly. Categories of continuous functors I. *Journal of Pure and Applied Algebra* Vol. 2, Issue 3, 169-191, 1972.

  as a context for proving reflectivity results for orthogonal subcategories and categories of models.

- The notion of locally bounded (symmetric monoidal closed) category was then explicitly defined by Kelly in [4, Chapter 6] and used as the basis for a general treatment of enriched limit theories.

Introduction

- Locally bounded categories subsume locally presentable categories and many "topological" categories that are *not* locally presentable.

- Speaking of locally presentable categories, in

  - [5] G.M. Kelly. Structures defined by finite limits in the enriched context I. *Cahiers de Topologie et Géométrie Catégoriques Différentielle* 23, No. 1, 3-42, 1982.

  Kelly defined the notion of a locally presentable $\mathscr{V}$-category over a locally presentable closed category $\mathscr{V}$.

- Kelly *did* define the notion of a locally bounded closed category $\mathscr{V}$ in [4, Chapter 6], but never got around to defining the notion of a locally bounded $\mathscr{V}$*-category* over such a $\mathscr{V}$. That's where this talk comes in!

Locally bounded (ordinary) categories

- Let's start by reviewing the definition of a locally bounded (ordinary) category. A **(proper) factegory** is a category $\mathscr{C}$ with a proper factorization system $(\mathscr{E}, \mathscr{M})$. The factegory $\mathscr{C}$ is **cocomplete** if $\mathscr{C}$ is cocomplete and has arbitrary cointersections (i.e. wide pushouts) of $\mathscr{E}$-morphisms.

- Given a small $\mathscr{M}$-family $(m_i : C_i \to C)_{i \in I}$ in $\mathscr{C}$, its **union** is the $\mathscr{M}$-subobject m obtained from the $(\mathscr{E}, \mathscr{M})$-factorization

$$\coprod_i C_i \xrightarrow{e} \bigcup_i C_i \xrightarrow{m} C.$$

  The family $(m_i)_i$ is α-**filtered** if any sub-family of size $< \alpha$ factors through some m_i.

Locally bounded (ordinary) categories

- A functor $U : \mathscr{C} \to \mathscr{D}$ between cocomplete factegories that preserves $\mathscr{M}$ is said to **preserve** (α-**filtered**) $\mathscr{M}$**-unions** if for any (α-filtered) $\mathscr{M}$-family $(m_i)_i$ with union m, Um is the union of the $\mathscr{M}$-family $(Um_i)_i$. If U preserves $\mathscr{M}$ and preserves α-filtered $\mathscr{M}$-unions, we also say that U is α-**bounded**.

- In particular, an object $C \in \mathbf{ob}\mathscr{C}$ of a cocomplete factegory $\mathscr{C}$ is α-bounded if $\mathscr{C}(C, -) : \mathscr{C} \to \mathbf{Set}$ preserves α-filtered $\mathscr{M}$-unions.

- Finally, a set $\mathscr{G} \subseteq \mathbf{ob}\mathscr{C}$ of a cocomplete factegory is an $(\mathscr{E}, \mathscr{M})$-**generator** if for any $C \in \mathbf{ob}\mathscr{C}$, the canonical morphism

$$\coprod_{G \in \mathscr{G}} \mathscr{C}(G, C) \cdot G \to C$$

lies in $\mathscr{E}$ (equivalently, the functors $\mathscr{C}(G, -) : \mathscr{C} \to \mathbf{Set}$ $(G \in \mathscr{G})$ are **jointly** $\mathscr{M}$**-conservative**).

Locally bounded (ordinary) categories

Definition (Kelly [4])

A locally α-bounded category is a cocomplete factegory $\mathscr{C}$ with an $(\mathscr{E}, \mathscr{M})$-generator consisting of α-bounded objects.

Note the parallel with locally α-presentable categories: a locally α-presentable category is a cocomplete category $\mathscr{C}$ with a *strong* generator consisting of α-*presentable* objects.

Examples

- Any locally α-presentable category [3, 3.2.3], with $(\mathscr{E}, \mathscr{M}) = (\mathbf{StrongEpi}, \mathbf{Mono})$ and the given strong generator of α-presentable (and hence α-bounded) objects.

- Any topological category over **Set** is locally $\aleph_0$-bounded [8, 2.3], with $(\mathscr{E}, \mathscr{M}) = (\mathbf{Epi}, \mathbf{StrongMono})$ and the generator consisting of just the discrete object on $\{*\}$.

- Any cocomplete locally cartesian closed category (e.g. elementary quasitopos) with a generator and arbitrary cointersections of epimorphisms, so that $(\mathscr{E}, \mathscr{M}) = (\mathbf{Epi}, \mathbf{StrongMono})$. These include the concrete quasitoposes of Dubuc [2].

Locally bounded closed categories

We now recall Kelly's definition of locally bounded symmetric monoidal closed category:

Definition (Kelly [4])

A symmetric monoidal closed category $\mathscr{V}$ is **locally α-bounded as a closed category** if $\mathscr{V}_0$ is locally α-bounded, the proper factorization system $(\mathscr{E}, \mathscr{M})$ is enriched, the unit object $I \in \mathbf{ob}\mathscr{V}$ is α-bounded, and $G \otimes G'$ is α-bounded for all $G, G' \in \mathscr{G}$.

For example: any symmetric monoidal closed category $\mathscr{V}$ with $\mathscr{V}_0$ locally α-presentable [4, Chapter 6]; any topological category over **Set**; any cocomplete locally cartesian closed category with generator and arbitrary epi-cointersections (e.g. any concrete quasitopos).

$\mathscr{V}$-factegories

- For the remainder of the talk, $\mathscr{V}$ will be a locally α-bounded closed category (sometimes a stronger assumption than needed).
- An enriched factorization system $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ on a $\mathscr{V}$-category $\mathscr{C}$ [7] is **compatible** with $(\mathscr{E}, \mathscr{M})$ if each $\mathscr{C}(C, -) : \mathscr{C} \to \mathscr{V}$ $(C \in \mathbf{ob}\mathscr{C})$ preserves the right class.
- A $\mathscr{V}$-**factegory** is a $\mathscr{V}$-category $\mathscr{C}$ with an enriched proper factorization system $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ that is compatible with $(\mathscr{E}, \mathscr{M})$. The $\mathscr{V}$-factegory $\mathscr{C}$ is **cocomplete** if the $\mathscr{V}$-category $\mathscr{C}$ is cocomplete and has arbitrary (conical) cointersections of $\mathscr{E}$-morphisms.

Enriched $(\mathscr{E}, \mathscr{M})$-generators

- Let $\mathscr{C}$ be a cocomplete $\mathscr{V}$-factegory. A set $\mathscr{G} \subseteq \mathbf{ob}\mathscr{C}$ is an **enriched $(\mathscr{E}, \mathscr{M})$-generator** if for each $C \in \mathbf{ob}\mathscr{C}$, the canonical morphism $\coprod_{G \in \mathscr{G}} \mathscr{C}(G, C) \otimes G \to C$ lies in $\mathscr{E}$.

- A set $\mathscr{G} \subseteq \mathbf{ob}\mathscr{C}$ is an enriched $(\mathscr{E}, \mathscr{M})$-generator iff the representable $\mathscr{V}$-functors $\mathscr{C}(G, -) : \mathscr{C} \to \mathscr{V}$ $(G \in \mathscr{G})$ are **jointly $\mathscr{M}$-conservative**.

Enriched α-bounded objects

Let $\mathscr{C}$ be a cocomplete $\mathscr{V}$-factegory. An object $C \in \mathbf{ob}\mathscr{C}$ is an **enriched α-bounded object** if $\mathscr{C}(C, -) : \mathscr{C} \to \mathscr{V}$ preserves α-filtered $\mathscr{M}$-unions.

Definition

A **locally α-bounded** $\mathscr{V}$-category is a cocomplete $\mathscr{V}$-factegory $\mathscr{C}$ with an enriched $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{G}$ consisting of enriched α-bounded objects.

- Note the parallel with locally α-presentable $\mathscr{V}$-categories: a locally α-presentable $\mathscr{V}$-category is a cocomplete $\mathscr{V}$-category with an enriched *strong* generator of enriched *α-presentable* objects [5, 3.1].
- If $\mathscr{V}$ is a locally α-bounded closed category with ordinary $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{G}$, then $\mathscr{V}$ is itself a locally α-bounded $\mathscr{V}$-category with enriched $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{G}$.
- Any locally bounded $\mathscr{V}$-category is total and complete.

Bounding right adjoints

The following notion of *bounding right adjoint* is fundamental for constructing examples of locally bounded $\mathscr{V}$-categories:

Definition

Let $U : \mathscr{C} \to \mathscr{D}$ be a $\mathscr{V}$-functor between cocomplete $\mathscr{V}$-factegories. Then U is an α-**bounding right adjoint** if U is α-bounded and has a left adjoint whose counit is pointwise in $\mathscr{E}$.

U is an α-bounding right adjoint iff U is α-bounded, has a left adjoint, and is $\mathscr{M}$-conservative, iff U is α-bounded, has a left adjoint, and reflects $\mathscr{E}$. An α-bounding right adjoint is automatically $\mathscr{V}$-faithful.

Bounding right adjoints

Theorem

Let $\mathscr{C}$ be a cocomplete $\mathscr{V}$-factegory and let $\mathscr{G} \subseteq \mathbf{ob}\mathscr{C}$ be a set. Then $\mathscr{C}$ is locally α-bounded with enriched $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{G}$ iff the nerve $\mathbf{y}_{\mathscr{G}} : \mathscr{C} \to [\mathscr{G}^{\mathbf{op}}, \mathscr{V}]$ is an α-bounding right adjoint.

Theorem

Let $\mathscr{D}$ be a locally α-bounded $\mathscr{V}$-category and $\mathscr{C}$ a cocomplete $\mathscr{V}$-factegory. If $U : \mathscr{C} \to \mathscr{D}$ is an α-bounding right adjoint, then $\mathscr{C}$ is locally α-bounded.

Bounding right adjoints

Theorem

Let $\mathscr{C}$ be a cocomplete $\mathscr{V}$-factegory. Then $\mathscr{C}$ is locally α-bounded iff there exists a small $\mathscr{V}$-category $\mathscr{A}$ and an α-bounding right adjoint $U : \mathscr{C} \to [\mathscr{A}, \mathscr{V}]$, i.e. a $\mathscr{V}$-functor $U : \mathscr{C} \to [\mathscr{A}, \mathscr{V}]$ that is α-bounded, $\mathscr{M}$-conservative, and has a left adjoint.

Note the parallel with Kelly's result [4, 3.1]: a cocomplete $\mathscr{V}$-category $\mathscr{C}$ is locally α-presentable iff there exists a small $\mathscr{V}$-category $\mathscr{A}$ and a $\mathscr{V}$-functor $U : \mathscr{C} \to [\mathscr{A}, \mathscr{V}]$ that has rank α, is conservative, and has a left adjoint.

Corollary: if $\mathscr{C}$ is locally α-bounded and $\mathscr{A}$ is small, then $[\mathscr{A}, \mathscr{C}]$ is locally α-bounded.

Enriched vs. ordinary local boundedness

Recall that $\mathscr{V}$ is a locally α-bounded closed category with ordinary $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{G}$.

Theorem

If $\mathscr{C}$ is locally α-bounded with enriched $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{H}$, then $\mathscr{C}_0$ is locally α-bounded with ordinary $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{G} \otimes \mathscr{H}$.

Theorem

If $\mathscr{C}$ is a cocomplete $\mathscr{V}$-factegory such that $\mathscr{C}_0$ is locally bounded with ordinary $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{H}$, then $\mathscr{C}$ is locally bounded with enriched $(\mathscr{E}, \mathscr{M})$-generator $\mathscr{H}$.

A representability theorem

It is well known that if $\mathscr{C}$ is a locally presentable (even accessible) category, then a functor $U : \mathscr{C} \to \mathbf{Set}$ is *representable* iff U is continuous and has rank. We have a similar result for locally bounded categories:

Theorem

Let $\mathscr{C}$ be a locally bounded and $\mathscr{E}$-cowellpowered $\mathscr{V}$-category. If $U : \mathscr{C} \to \mathscr{V}$ preserves $\mathscr{M}$, then U is representable iff U is continuous and bounded.

Adjoint functor theorems

Recall that a functor $U : \mathscr{C} \to \mathscr{D}$ between locally presentable categories has a left adjoint iff U is continuous and has rank.

Theorem

Let $\mathscr{C}, \mathscr{D}$ be locally bounded $\mathscr{V}$-categories such that $\mathscr{C}$ is $\mathscr{E}$-cowellpowered. If $U : \mathscr{C} \to \mathscr{D}$ preserves $\mathscr{M}$, then U has a left adjoint iff U is continuous and bounded.

Recall that if $\mathscr{C}$ is locally presentable and $\mathscr{D}$ arbitrary, then $F : \mathscr{C} \to \mathscr{D}$ has a right adjoint iff F is cocontinuous.

Theorem

Let $F : \mathscr{C} \to \mathscr{D}$ be a $\mathscr{V}$-functor from a locally bounded $\mathscr{V}$-category $\mathscr{C}$ to an arbitrary $\mathscr{V}$-category $\mathscr{D}$. Then F has a right adjoint iff F is cocontinuous.

(In fact, $\mathscr{C}$ just needs to be cocomplete $\mathscr{V}$-factegory with enriched $(\mathscr{E}, \mathscr{M})$-generator.)

α-bounded-small limits

- It is well known that α-small limits commute with α-filtered colimits in any locally α-presentable category.
- If $\mathscr{V}$ is a locally α-presentable closed category, then Kelly defined in [5, 4.1] the notion of an *α-small weight* $W : \mathscr{B} \to \mathscr{V}$: $|\mathbf{ob}\mathscr{B}| < \alpha$, $\mathscr{B}(B, B') \in \mathscr{V}_\alpha$ for all $B, B' \in \mathbf{ob}\mathscr{B}$, and $WB \in \mathscr{V}_\alpha$ for all $B \in \mathbf{ob}\mathscr{B}$.
- He then showed in [5, 4.9] that α-small weighted limits commute with conical α-filtered colimits in any locally α-presentable $\mathscr{V}$-category.
- If $\mathscr{V}$ is a locally α-bounded closed category, we can define the similar notion of an *α-bounded-small* weight $W : \mathscr{B} \to \mathscr{V}$.

α-bounded-small limits

Definition

Let $\mathscr{V}$ be a locally α-bounded closed category. A weight $W : \mathscr{B} \to \mathscr{V}$ is α-**bounded-small** if $|\mathbf{ob}\mathscr{B}| < \alpha$, $\mathscr{B}(B, B')$ is an enriched α-bounded object of $\mathscr{V}$ for all $B, B' \in \mathbf{ob}\mathscr{B}$, and WB is an enriched α-bounded object of $\mathscr{V}$ for all $B \in \mathbf{ob}\mathscr{B}$.

Kelly showed in [5, 4.3] that the saturation of the class of α-small weights is equal to the saturation of the class of weights for α-small conical limits and α-presentable cotensors. We similarly have:

Theorem

The saturation of the class of α-bounded-small weights is equal to the saturation of the class of weights for α-small conical limits and α-bounded cotensors.

α-bounded-small limits

Definition

Let $\mathscr{C}$ be a complete and cocomplete $\mathscr{V}$-factegory and $W : \mathscr{B} \to \mathscr{V}$ a small weight. Then W-**limits commute with** α-**filtered** $\mathscr{M}$-**unions in** $\mathscr{C}$ if the W-limit $\mathscr{V}$-functor $\{W, -\} : [\mathscr{B}, \mathscr{C}] \to \mathscr{C}$ is α-bounded.

Theorem

If $\mathscr{C}$ is a locally α-bounded $\mathscr{V}$-category, then α-bounded-small weighted limits commute with α-filtered $\mathscr{M}$-unions in $\mathscr{C}$.

Reflectivity and local boundedness

- Freyd and Kelly proved in [3, 4.1.3, 4.2.2] that if $\mathscr{C}$ is an $\mathscr{E}$-cowellpowered locally bounded ordinary category and Θ is a "quasi-small" class of morphisms in $\mathscr{C}$, then the orthogonal subcategory $\Theta^{\perp} \hookrightarrow \mathscr{C}$ is reflective and locally bounded.

- Kelly showed in [4, Chapter 6] that the reflectivity still holds even without $\mathscr{E}$-cowellpoweredness.

We have enriched both results as follows:

Theorem

Let $\mathscr{C}$ be a locally bounded $\mathscr{V}$-category with a "quasi-small" class of morphisms Θ. Then the enriched orthogonal sub-$\mathscr{V}$-category $\Theta^{\perp_{\mathscr{V}}} \hookrightarrow \mathscr{C}$ is reflective, and $\Theta^{\perp_{\mathscr{V}}}$ is locally bounded if $\mathscr{C}$ is $\mathscr{E}$-cowellpowered.

Reflectivity and local boundedness

Freyd and Kelly also proved in [3, 5.2.1, 5.2.2] that if $\mathscr{C}$ is a locally bounded and $\mathscr{E}$-cowellpowered ordinary category and $(\mathscr{A}, \Phi)$ is a limit sketch, then Φ-**Mod**$(\mathscr{A}, \mathscr{C})$ is reflective in $[\mathscr{A}, \mathscr{C}]$ and locally bounded.

Theorem

*Let $\mathscr{C}$ be a locally α-bounded $\mathscr{V}$-category and $(\mathscr{A}, \Phi)$ an enriched limit sketch [4, 6.3]. Then the full sub-$\mathscr{V}$-category Φ-**Mod**$(\mathscr{A}, \mathscr{C}) \hookrightarrow [\mathscr{A}, \mathscr{C}]$ is reflective, and Φ-**Mod**$(\mathscr{A}, \mathscr{C})$ is also locally bounded if $\mathscr{C}$ is $\mathscr{E}$-cowellpowered. If every weight in Φ is α-bounded-small, then Φ-**Mod**$(\mathscr{A}, \mathscr{C})$ is in fact locally α-**bounded**.*

*In particular, if $\mathscr{T}$ is a Φ-theory for a class of small weights Φ, then Φ-**Cts**$(\mathscr{T}, \mathscr{C})$ is reflective in $[\mathscr{T}, \mathscr{C}]$, and is locally bounded if $\mathscr{C}$ is $\mathscr{E}$-cowellpowered.*

Reflectivity and local boundedness

As a corollary, we obtain the following result for the enriched algebraic theories of Lucyshyn-Wright [6]:

Theorem

Let $\mathscr{J} \hookrightarrow \mathscr{V}$ be a small system of arities, let $\mathscr{T}$ be a $\mathscr{J}$-theory, and let $\mathscr{C}$ be a locally bounded and $\mathscr{E}$-cowellpowered $\mathscr{V}$-category. Then the full sub-$\mathscr{V}$-category $\mathscr{T}$-$\mathbf{Alg}(\mathscr{C}) \hookrightarrow [\mathscr{T}, \mathscr{C}]$ of the $\mathscr{T}$-algebras is reflective and locally bounded, and the forgetful $\mathscr{V}$-functor $U^{\mathscr{T}} : \mathscr{T}$-$\mathbf{Alg}(\mathscr{C}) \to \mathscr{C}$ is monadic.

In particular, if $\mathscr{C}$ is a locally α-bounded and $\mathscr{E}$-cowellpowered ordinary category and $\mathscr{T}$ is a Lawvere theory, then the category $\mathscr{T}$-$\mathbf{Alg}(\mathscr{C})$ of $\mathscr{T}$-algebras in $\mathscr{C}$ is reflective in $[\mathscr{T}, \mathscr{C}]$ and locally α-bounded, and the forgetful functor $U^{\mathscr{T}} : \mathscr{T}$-$\mathbf{Alg}(\mathscr{C}) \to \mathscr{C}$ is monadic.

In summary...

- We have defined a notion of locally bounded $\mathscr{V}$-category over a locally bounded closed category $\mathscr{V}$, which enriches the locally bounded ordinary categories of Freyd and Kelly, and parallels Kelly's notion of locally presentable $\mathscr{V}$-category over a locally presentable closed category $\mathscr{V}$.
- Examples of locally bounded closed categories include locally presentable closed categories, topological categories over **Set**, and epi-cocomplete quasitoposes with generators.
- Many of the results for locally presentable enriched categories have analogues for locally bounded enriched categories: representability theorems, adjoint functor theorems, and commutation of suitably small limits with suitably filtered colimits/unions.

In summary...

- Moreover, locally bounded enriched categories admit full enrichments of Freyd and Kelly's reflectivity results for orthogonal subcategories and categories of models.
- Lucyshyn-Wright and I have also shown that locally bounded enriched categories provide a fruitful setting for obtaining results on free monads, presentations of monads, and algebraic colimits of monads for a subcategory of arities. I will talk about this at another ATCAT seminar!

α-bounded monads?

- One topic I did *not* touch on is the local boundedness of Eilenberg-Moore categories. It is (well) known that if $\mathscr{C}$ is a locally α-presentable $\mathscr{V}$-category and $\mathbb{T}$ is a $\mathscr{V}$-monad on $\mathscr{C}$ with rank α, then the Eilenberg-Moore $\mathscr{V}$-category $\mathbb{T}$-**Alg** is locally α-presentable, and $U^{\mathbb{T}} : \mathbb{T}\text{-}\mathbf{Alg} \to \mathscr{C}$ is continuous and has rank α (see [1, 6.9]).

- Does an analogous result hold for α-bounded $\mathscr{V}$-monads on locally α-bounded $\mathscr{V}$-categories? Essentially yes, but with some slight subtleties/complications (to be presented in forthcoming work).

Thank you!

References I

[1] G. Bird. Limits in 2-Categories of Locally-Presented Categories. PhD thesis, University of Sydney, 1984.

[2] E.J. Dubuc. Concrete quasitopoi. In: M. Fourman, C. Mulvey, D. Scott (eds.) *Applications of Sheaves*. Lecture Notes in Mathematics, Vol. 753. Springer, Berlin, Heidelberg, 1979.

[3] P.J. Freyd and G.M. Kelly. Categories of continuous functors I. *Journal of Pure and Applied Algebra* Vol. 2, Issue 3, 169-191, 1972.

[4] G.M. Kelly. Basic concepts of enriched category theory. *Repr. Theory and Applications of Categories*, No. 10, 2005, Reprint of the 1982 original [Cambridge University Press].

[5] G.M. Kelly. Structures defined by finite limits in the enriched context I. *Cahiers de Topologie et Géométrie Catégoriques Différentielle* 23, No. 1, 3-42, 1982.

References II

[6] R.B.B. Lucyshyn-Wright. Enriched algebraic theories and monads for a system of arities. *Theory and Applications of Categories* Vol. 31, No. 5, 101-137, 2016.

[7] R.B.B. Lucyshyn-Wright. Enriched factorization systems. *Theory and Applications of Categories* Vol. 29, No. 18, 475-495, 2014.

[8] L. Sousa. On boundedness and small-orthogonality classes. *Cahiers de Topologie et Géométrie Catégoriques Différentielle* 50, No. 1, 67-79, 2009.