Isotropy in Category Theory

Jason Parker

Postdoctoral Fellow
Department of Mathematics and Computer Science, Brandon University

Brandon University Science Seminar
March 25, 2021

Introduction

- **Isotropy** is a (new) mathematical phenomenon with manifestations in category theory, algebra, and theoretical computer science.

- We will see that isotropy encodes a generalized notion of *symmetry* for many prominent algebraic structures in mathematics.

Sets

- A *set* is a collection of objects (finite or infinite).
- E.g. the set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ of *integers*.
- E.g. the set $\mathbb{Q}$ of *rational numbers*.

Functions

Given two sets X and Y, a *function* $\alpha : X \to Y$ is a rule that assigns to every element $x \in X$ a unique element $\alpha(x) \in Y$. A function $\alpha : X \to Y$ is *bijective* if for every $y \in Y$, there is a *unique* $x \in X$ with $\alpha(x) = y$.

Permutations

Given a set X, one can form the set $\mathbf{Perm}(X)$ of all bijective functions $\alpha : X \to X$, i.e. all *permutations* or *symmetries* of the set X.

Permutations

The set $\mathbf{Perm}(X)$ comes equipped with operations of *product*, *inverse*, and *identity*. First, any two permutations of X can be *composed*: given $\alpha, \beta \in \mathbf{Perm}(X)$, we can define $\alpha \circ \beta \in \mathbf{Perm}(X)$.

Permutations

There is an *identity* permutation **id** of X, with $\alpha \circ \mathbf{id} = \alpha = \mathbf{id} \circ \alpha$ for any $\alpha \in \mathbf{Perm}(X)$:

Permutations

Any permutation α of X has an *inverse* permutation α^{-1} of X with $\alpha \circ \alpha^{-1} = \textbf{id} = \alpha^{-1} \circ \alpha$:

Groups

- The set $\mathbf{Perm}(X)$, together with the operations $\circ, \mathbf{id}, (-)^{-1}$ of product, identity, and inverse form an algebraic structure called a *group*.

- For another example, consider the set $\mathbb{Z}$ of integers. Any two integers $n, m \in \mathbb{Z}$ have a 'product' $n + m \in \mathbb{Z}$. There is an 'identity' integer $0 \in \mathbb{Z}$, with $n + 0 = n = 0 + n$ for every integer $n \in \mathbb{Z}$. Every integer n has an 'inverse' integer $-n$ with $n + (-n) = 0 = (-n) + n$.

- So the set $\mathbb{Z}$ together with the operations $+, 0, -$ also forms a *group*.

Formal Definition of Groups

A *group* is a set G with operations of product $\cdot$, identity e, and inverse $^{-1}$. So $e \in G$, any two elements $g, h \in G$ have a product $g \cdot h \in G$, and any element $g \in G$ has an inverse element $g^{-1} \in G$. These data must satisfy the following axioms:

- $g \cdot (h \cdot k) = (g \cdot h) \cdot k$ for any $g, h, k \in G$.
- $g \cdot e = g = e \cdot g$ for any $g \in G$.
- $g \cdot g^{-1} = e = g^{-1} \cdot g$ for any $g \in G$.

Group Homomorphisms

- If G and H are groups, then a function $f : G \to H$ is called a *group homomorphism* if f preserves the product operation:

$$f(g \cdot h) = f(g) \cdot f(h)$$

  for any $g, h \in G$.

- E.g. the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2n$ is a group homomorphism, because

$$f(n + m) = 2(n + m) = 2n + 2m = f(n) + f(m)$$

  for any $n, m \in \mathbb{Z}$.

- A group homomorphism $\alpha : G \to G$ which is also a permutation is called a group *automorphism* of G.

Inner Automorphisms

- Let G be a group. Any element $g \in G$ induces an automorphism $\mathbf{inn}_g : G \to G$ by 'conjugation' with g, i.e.

  $$\mathbf{inn}_g(h) := g \cdot h \cdot g^{-1}$$

  for any $h \in G$. An automorphism of this kind is called an *inner automorphism*.

- E.g. if $\mathbf{Perm}(X)$ is the group of permutations on a set X and $\alpha : X \to X$ is a permutation, then we have a group automorphism $\mathbf{inn}_\alpha : \mathbf{Perm}(X) \to \mathbf{Perm}(X)$ defined by

  $$\mathbf{inn}_\alpha(\beta) := \alpha \circ \beta \circ \alpha^{-1}$$

  for any $\beta \in \mathbf{Perm}(X)$.

Extendibility of Inner Automorphisms

Let G be a group and $\mathbf{inn}_g : G \to G$ an inner automorphism of G. If $f : G \to H$ is a group homomorphism to another group H, then we can *extend* or *push* $\mathbf{inn}_g$ *forward* along f to get another inner automorphism $\mathbf{inn}_{f(g)}$ of H:

Bergman's Result

- In [1], George Bergman essentially proved a converse result. Namely, if $\alpha : G \to G$ is a group automorphism that can be extended along any group homomorphism $f : G \to H$ to another group H, then α *must be* an inner automorphism.

- This gives a completely *abstract* or *categorical* formulation of the notion of *inner automorphism* in group theory: a group automorphism $\alpha : G \to G$ is *inner* if and only if α can be extended along any group homomorphism $f : G \to H$.

- This formulation of *inner automorphism* also makes sense for any other kind of algebraic structure, and even for any *category*.

Universal Algebra

- In the field of *universal algebra*, there is a general way to define arbitrary kinds of algebraic structures, i.e. sets equipped with operations satisfying certain equations.
- Given two algebras M, N of the same kind, a function $f : M \to N$ is called a *homomorphism* if it preserves the algebraic operations.
- A homomorphism $\alpha : M \to M$ of an algebra M is called an *automorphism* if it is also a permutation of M.

Isotropy of Algebraic Structures

- Let M be an algebra. Turning Bergman's result for groups into a definition, we say that an automorphism $\alpha : M \to M$ is *inner* if it can be extended along any homomorphism $f : M \to N$ to another algebra N of the same kind.

- We then define $\mathcal{Z}(M)$, the *isotropy group* of M, to be the group of all such inner automorphisms of M.

- In my research ([3, 4]), I have given concrete and explicit characterizations of $\mathcal{Z}(M)$ for many different kinds of algebraic structures: monoids, groups, rings, modules, group actions, and more general structures in *category theory*.

Isotropy in Category Theory

- In full generality, any category $\mathbb{C}$ has an isotropy group *functor* $\mathcal{Z} : \mathbb{C} \to \mathbf{Group}$ that sends any object $C \in \mathbb{C}$ to its group of *inner automorphisms*.

- This is a category-theoretic *invariant* (first introduced in [2]) that captures information about the behaviour of automorphisms in a category.

- My research has completely characterized the isotropy group functors of a wide class of categories, including any category of (essentially) algebraic structures. Therefore, I have shown how to equip any such category with a notion of *inner automorphism*.

Thank you!

References

- G. Bergman. An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange. Publicacions Matematiques 56, 91-126, 2012.
- J. Funk, P. Hofstra, B. Steinberg. Isotropy and crossed toposes. Theory and Applications of Categories 26, 660-709, 2012.
- P. Hofstra, J. Parker, P.J. Scott. Isotropy of algebraic theories. Electronic Notes in Theoretical Computer Science 341, 201-217, 2018.
- J. Parker. Isotropy Groups of Quasi-Equational Theories. PhD thesis, University of Ottawa, 2020.