Diagrammatic presentations of enriched monads and the axiomatics of enriched algebra

Jason Parker
(joint work with Rory Lucyshyn-Wright)

Natural Sciences and Engineering Research Council of Canada
Conseil de recherches en sciences naturelles et en génie du Canada
Canada

Quick summary

Given a suitable **subcategory of arities** $\mathcal{J} \hookrightarrow \mathcal{C}$ in a suitable $\mathcal{V}$-category $\mathcal{C}$ enriched in a suitable symmetric monoidal closed category $\mathcal{V}$, we establish flexible and user-friendly methods for presenting **enriched $\mathcal{J}$-algebraic structure** in a wide variety of (new) mathematical contexts, and we establish that the applicability of such methods is equivalent to the axiom of **strong amenability** introduced in [LWP23b] and discussed in Rory Lucyshyn-Wright's talk. This work generalizes and extends previous work in [KP93, BG19, LWP22b, LWP23a].

Strongly amenable subcategories of arities

- Fix a complete and cocomplete symmetric monoidal closed category $\mathcal{V}$, a complete and cocomplete $\mathcal{V}$-category $\mathcal{C}$, and a small **subcategory of arities** $\mathcal{J} \hookrightarrow \mathcal{C}$, i.e. a small dense full sub-$\mathcal{V}$-category.
- A **$\mathcal{J}$-pretheory** is a $\mathcal{V}$-category $\mathcal{T}$ equipped with an identity-on-objects $\mathcal{V}$-functor $\tau : \mathcal{J}^{\mathrm{op}} \to \mathcal{T}$. There is a $\mathcal{V}$-category $\mathcal{T}$-Alg of **$\mathcal{T}$-algebras** equipped with a forgetful $\mathcal{V}$-functor $U^{\mathcal{T}} : \mathcal{T}\text{-Alg} \to \mathcal{C}$.
- The subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$ is **strongly amenable** [LWP23b] if $U^{\mathcal{T}} : \mathcal{T}\text{-Alg} \to \mathcal{C}$ has a left adjoint for every $\mathcal{J}$-pretheory $\mathcal{T}$ (briefly: if every $\mathcal{J}$-pretheory has free algebras). This will be discussed further in Rory Lucyshyn-Wright's talk.

Free-form $\mathcal{J}$-signatures and diagrammatic $\mathcal{J}$-presentations [LWP23a]

- A **parametrized ($\mathcal{J}$-)operation** on an object A of $\mathcal{C}$ is a morphism $\omega : \mathcal{C}(J, A) \to \mathcal{C}(C, A)$ in $\mathcal{V}$ for objects $J \in \mathrm{ob}\,\mathcal{J}$ (the **arity**) and $C \in \mathrm{ob}\,\mathcal{C}$ (the **parameter**). Equivalently, we can write $\omega : \mathcal{C}(J, A) \otimes C \to A$ in $\mathcal{C}$, because $\mathcal{C}$ is tensored.
- A **free-form $\mathcal{J}$-signature** is a set $\mathcal{S}$ (of **operation symbols**) equipped with objects $J_\sigma \in \mathrm{ob}\,\mathcal{J}$, $C_\sigma \in \mathrm{ob}\,\mathcal{C}$ ($\sigma \in \mathcal{S}$). An **$\mathcal{S}$-algebra** is an object A of $\mathcal{C}$ equipped with parametrized operations $\sigma^A : \mathcal{C}\left(J_\sigma, A\right) \to \mathcal{C}\left(C_\sigma, A\right)$ ($\sigma \in \mathcal{S}$). There is a $\mathcal{V}$-category $\mathcal{S}$-Alg of $\mathcal{S}$-algebras equipped with a forgetful $\mathcal{V}$-functor $U^{\mathcal{S}} : \mathcal{S}\text{-Alg} \to \mathcal{C}$.
- A **diagrammatic $\mathcal{S}$-equation** $\omega \doteq \nu$ is a pair of $\mathcal{V}$-natural transformations $\omega, \nu : \mathcal{C}\left(J, U^{\mathcal{S}}-\right) \Longrightarrow \mathcal{C}\left(C, U^{\mathcal{S}}-\right) : \mathcal{S}\text{-Alg} \to \mathcal{V}$ with $J \in \mathrm{ob}\,\mathcal{J}$ and $C \in \mathrm{ob}\,\mathcal{C}$. An $\mathcal{S}$-algebra A **satisfies** a diagrammatic $\mathcal{S}$-equation $\omega \doteq \nu$ if $\omega_A = \nu_A : \mathcal{C}(J, A) \to \mathcal{C}(C, A)$.
- A **diagrammatic $\mathcal{J}$-presentation** is a pair $\mathcal{P} = (\mathcal{S}, \mathcal{E})$ consisting of a free-form $\mathcal{J}$-signature $\mathcal{S}$ and a small family $\mathcal{E}$ of diagrammatic $\mathcal{S}$-equations. A **$\mathcal{P}$-algebra** is an $\mathcal{S}$-algebra that satisfies the diagrammatic $\mathcal{S}$-equations in $\mathcal{E}$. We write $\mathcal{P}\text{-Alg} \hookrightarrow \mathcal{S}\text{-Alg}$ for the full sub-$\mathcal{V}$-category consisting of the $\mathcal{P}$-algebras, and $U^{\mathcal{P}} : \mathcal{P}\text{-Alg} \to \mathcal{C}$ for the restricted forgetful $\mathcal{V}$-functor, so that we may regard $\mathcal{P}$-Alg as an object of the slice category $\mathcal{V}\text{-CAT}/\mathcal{C}$.
- A diagrammatic $\mathcal{J}$-presentation $\mathcal{P}$ **presents a $\mathcal{V}$-monad (on $\mathcal{C}$)** if there is a $\mathcal{V}$-monad $\mathbb{T}_{\mathcal{P}}$ on $\mathcal{C}$ (necessarily unique up to isomorphism) such that $\mathcal{P}\text{-Alg} \cong \mathbb{T}_{\mathcal{P}}\text{-Alg}$ in $\mathcal{V}\text{-CAT}/\mathcal{C}$. A **$\mathcal{J}$-ary variety** is an object of $\mathcal{V}\text{-CAT}/\mathcal{C}$ of the form $\mathcal{P}$-Alg for some diagrammatic $\mathcal{J}$-presentation $\mathcal{P}$.

The main result

Theorem: The small subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$ is strongly amenable if and only if every diagrammatic $\mathcal{J}$-presentation presents a $\mathcal{V}$-monad on $\mathcal{C}$.

- This theorem establishes that strongly amenable subcategories of arities enable flexible and user-friendly methods for presenting enriched algebraic structure, and that the axiom of strong amenability is *equivalent* to the requirement that every diagrammatic presentation present a $\mathcal{V}$-monad on $\mathcal{C}$.
- In this situation, the $\mathcal{V}$-monads on $\mathcal{C}$ that have diagrammatic $\mathcal{J}$-presentations are precisely the $\mathcal{J}$-**nervous** $\mathcal{V}$-monads of [LWP23b], and the category $\mathrm{Mnd}_{\mathcal{J}}(\mathcal{C})$ of $\mathcal{J}$-nervous $\mathcal{V}$-monads on $\mathcal{C}$ is monadic over the category $\mathrm{Sig}_{\mathcal{J}}(\mathcal{C})$ of $\mathcal{J}$-signatures, and is dually equivalent to the category $\mathrm{Var}_{\mathcal{J}}(\mathcal{C})$ of $\mathcal{J}$-ary varieties.

Examples of strongly amenable subcategories of arities

- Every **bounded** and **eleutheric** [LWP22b] subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$. In particular, every small eleutheric subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$ in a **locally bounded $\mathcal{V}$-category** $\mathcal{C}$ [LWP22a] enriched in a **locally bounded closed category** $\mathcal{V}$ [Kel05].
- Every **accessible** subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$ (i.e. there is a regular cardinal α such that each $\mathcal{C}(J, -) : \mathcal{C} \to \mathcal{V}$ ($J \in \mathrm{ob}\,\mathcal{J}$) preserves small conical α-filtered colimits).
- Every small subcategory of arities $\mathcal{J} \hookrightarrow \mathcal{C}$ in a **$\mathcal{V}$-sketchable** $\mathcal{V}$-category $\mathcal{C}$ (i.e. $\mathcal{C} \simeq \Phi\text{-Cts}(\mathfrak{T}, \mathcal{V})$ for a class of small weights Φ and a small Φ-theory $\mathfrak{T}$) enriched in a locally bounded closed category $\mathcal{V}$. In particular, every small full sub-$\mathcal{V}$-category $\mathcal{J} \hookrightarrow \mathcal{V}$ that contains the unit object I in a locally bounded closed category $\mathcal{V}$ (e.g. any topological category over **Set**), and every small subcategory of arities in a locally presentable $\mathcal{V}$-category $\mathcal{C}$ enriched in a locally presentable closed category $\mathcal{V}$ [BG19].

Some examples of $\mathcal{J}$-ary varieties

- With $\mathcal{C} = \mathcal{V}$ cartesian closed and $\mathcal{J} = \{n \cdot I \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$: **internal R-modules** in $\mathcal{V}$ for an **internal rig** R in $\mathcal{V}$.
- With $\mathcal{C} = \mathcal{V} = \mathbf{Cat}$ and $\mathcal{J} = \{n \cdot 1 \mid n \in \mathbb{N}\} \hookrightarrow \mathbf{Cat}$: small monoidal categories.
- With $\mathcal{V}$ locally bounded and $\mathcal{C} = \mathbf{Grph}(\mathcal{V})$ (**internal graphs** in $\mathcal{V}$): **internal categories** in $\mathcal{V}$.
- With $\mathcal{V}$ cartesian closed and locally bounded and $\mathcal{J} \hookrightarrow \mathcal{V}$ any small **system of arities**: the (enriched) **global state algebras** of [PP02].
- With $\mathcal{V}$ cartesian closed and $\mathcal{J} = \{n \cdot 1 \mid n \in \mathbb{N}\} \hookrightarrow \mathcal{V}$: algebras for Staton's parametrized theory of **instantiating and reading bits** [Sta13].
- With $\mathcal{C} = \mathcal{V} = \mathbf{CGTop}$ and $\mathcal{J} = \{n \cdot 1 \mid n \in \mathbb{N}\} \hookrightarrow \mathbf{CGTop}$: **H-spaces**, i.e. internal 'monoids' in **CGTop** whose multiplication is only associative and unital up to specified homotopies.

References

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