Presentations and Algebraic Colimits of Enriched Monads

Rory Lucyshyn-Wright and Jason Parker

Brandon University, Manitoba, Canada

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Introduction

- Signatures and presentations for monads and theories in (enriched) universal algebra have been previously studied mainly in the context of *locally presentable* enriched categories; see e.g.
  - [5] G.M. Kelly and A.J. Power. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra* Vol. 89 (1993) 163-179.
  - [6] S. Lack. On the monadicity of finitary monads. *Journal of Pure and Applied Algebra* Vol. 140 (1999) 65-73.
  - [3] J. Bourke and R. Garner. Monads and theories. *Advances in Mathematics* Vol. 351 (2019) 1024-1071.
- In this talk, we discuss recent work on developing a framework for such phenomena that subsumes a much wider class of enriched categories, including *locally bounded* enriched categories and the symmetric monoidal closed π-*categories* of [2].

Subcategories of Arities

- A *subcategory of arities* $j : \mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ is a full and dense sub-$\mathscr{V}$-category. In this talk we will also assume $\mathscr{J}$ to be small.

- A subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is *eleutheric* ([8]) if any $\mathscr{V}$-functor $F : \mathscr{J} \to \mathscr{C}$ has a left Kan extension along j, which is moreover preserved by $\mathscr{C}(J, -) : \mathscr{C} \to \mathscr{V}$ for each $J \in \mathbf{ob}\,\mathscr{J}$.

- This entails that $\mathscr{V}\mathbf{CAT}(\mathscr{J}, \mathscr{C}) \simeq \mathbf{End}_{\mathscr{J}}(\mathscr{C})$, the ordinary category of $\mathscr{J}$*-ary* $\mathscr{V}$-endofunctors of $\mathscr{C}$, i.e. $\mathscr{V}$-endofunctors $H : \mathscr{C} \to \mathscr{C}$ that preserve left Kan extensions along j.

$\mathscr{J}$-ary Monads and $\mathscr{J}$-Theories

- If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is a subcategory of arities, then $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ is the (ordinary) category of $\mathscr{J}$-ary $\mathscr{V}$-monads on $\mathscr{C}$, i.e. $\mathscr{V}$-monads with underlying $\mathscr{J}$-ary endofunctor.

- If $\mathscr{C} = \mathscr{V}$ and $j : \mathscr{J} \hookrightarrow \mathscr{V}$ is an eleutheric *system* of arities (i.e. $\mathscr{J}$ contains I and is closed under $\otimes$), then $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{V}) \simeq \mathbf{Th}_{\mathscr{J}}$, the category of $\mathscr{J}$*-theories* ([8]).

- We will now discuss the assumptions that we will impose on our subcategories of arities, and then give examples.

Factegories

- A *symmetric monoidal closed factegory* is a symmetric monoidal closed category $\mathscr{V}$ with an enriched factorization system $(\mathscr{E}, \mathscr{M})$ ([9]).
- If $\mathscr{C}$ is a $\mathscr{V}$-category over a symmetric monoidal closed factegory $\mathscr{V}$, then an enriched factorization system $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ on $\mathscr{C}$ is *compatible* with $(\mathscr{E}, \mathscr{M})$ if for any $C \in \mathbf{ob}\mathscr{C}$, the $\mathscr{V}$-functor $\mathscr{C}(C, -) : \mathscr{C} \to \mathscr{V}$ preserves the right class.
- If $\mathscr{V}$ is a closed factegory, then a *$\mathscr{V}$-factegory* is a $\mathscr{V}$-category $\mathscr{C}$ equipped with an enriched factorization system $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ that is compatible with $(\mathscr{E}, \mathscr{M})$.
- A $\mathscr{V}$-factegory $\mathscr{C}$ is *cocomplete* if $\mathscr{C}$ is cocomplete and has arbitrary cointersections of $\mathscr{E}_{\mathscr{C}}$-morphisms.

Boundedness

- Let α be a regular cardinal and let $\mathscr{C}$ and $\mathscr{D}$ be $\mathscr{V}$-factegories with conical α-filtered colimits. A $\mathscr{V}$-functor $F : \mathscr{C} \to \mathscr{D}$ is *α-bounded* if for any α-filtered diagram $D : \mathscr{A} \to \mathscr{C}_0$ with colimit $\mathbf{colim} D$ and any $\mathscr{M}_{\mathscr{C}}$-cocone $m = (m_A : D_A \to C)_A$ on D, if $\overline{m} : \mathbf{colim} D \to C$ lies in $\mathscr{E}_{\mathscr{C}}$, then $\overline{Fm} : \mathbf{colim} FD \to FC$ lies in $\mathscr{E}_{\mathscr{D}}$.

- We say that $F : \mathscr{C} \to \mathscr{D}$ is *bounded* if F is α-bounded for some α. If $C \in \mathbf{ob}\mathscr{C}$, then C is *α-bounded* if the $\mathscr{V}$-functor $\mathscr{C}(C, -) : \mathscr{C} \to \mathscr{V}$ is α-bounded.

- If $(\mathscr{E}, \mathscr{M}) = (\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}}) = (\mathscr{E}_{\mathscr{D}}, \mathscr{M}_{\mathscr{D}}) = (\mathbf{Iso}, \mathbf{All})$, then F is α-bounded iff F preserves α-filtered colimits.

Boundedness

If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is a subcategory of arities in a $\mathscr{V}$-factegory $\mathscr{C}$, then $\mathscr{J}$ is (α-)*bounded* if every $J \in \mathbf{ob}\,\mathscr{J}$ is (α-)bounded.

Proposition

If $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is an α-bounded subcategory of arities in a cocomplete $\mathscr{V}$-factegory $\mathscr{C}$, then any $\mathscr{J}$-ary endofunctor $H : \mathscr{C} \to \mathscr{C}$ is α-bounded.

Blanket Assumptions

All of our results hold for any subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$ satisfying the following assumptions:

- $\mathscr{V}$ is a complete and cocomplete symmetric monoidal closed factegory.
- $\mathscr{C}$ is a cocomplete and cotensored $\mathscr{V}$-factegory such that either $(\mathscr{E}_{\mathscr{C}}, \mathscr{M}_{\mathscr{C}})$ is *proper* or $\mathscr{C}$ is $\mathscr{E}_{\mathscr{C}}$-cowellpowered.
- $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is bounded and eleutheric.

Examples

- The subcategory of arities $j : \mathscr{C}_\alpha \hookrightarrow \mathscr{C}$ of α-presentable objects in a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally α-presentable closed category $\mathscr{V}$. The $\mathscr{J}$-ary endofunctors are those that preserve α-filtered colimits (i.e. have rank α); the $\mathscr{J}$-ary monads correspond to the enriched Lawvere theories of [12] for $\alpha = \aleph_0$.

- The subcategory of arities $j : \mathscr{C}_\Phi \hookrightarrow \mathscr{C}$ consisting of suitable Φ-presentable objects in a locally Φ-presentable $\mathscr{V}$-category for a class of weights Φ satisfying Axiom A from [7] in a locally bounded and $\mathscr{E}$-cowellpowered $\mathscr{V}$. The $\mathscr{J}$-ary endofunctors are those that preserve Φ-flat colimits; the $\mathscr{J}$-ary monads correspond to the Lawvere Φ-theories of [7].

Examples

- In particular, the subcategory of arities $j : \mathscr{C}_{\mathbb{D}} \hookrightarrow \mathscr{C}$ of $\mathbb{D}$-presentable objects in a locally $\mathbb{D}$-presentable $\mathscr{V}$-category over a locally $\mathbb{D}$-presentable closed category $\mathscr{V}$ for a sound doctrine $\mathbb{D}$ ([1]). The $\mathscr{J}$-ary endofunctors are those that preserve conical $\mathbb{D}$-filtered colimits.

- The subcategory of arities $j : \{I\} \hookrightarrow \mathscr{V}$ consisting of just the unit object in a complete and cocomplete symmetric monoidal closed factegory $\mathscr{V}$. The $\mathscr{J}$-ary endofunctors are those of the form $X \otimes (-) : \mathscr{V} \to \mathscr{V}$; the $\mathscr{J}$-ary monads correspond to monoids in $\mathscr{V}$.

Examples

- The subcategory of arities $j : \mathbb{N}_{\mathscr{V}} \hookrightarrow \mathscr{V}$ consisting of the finite copowers of the unit object in any symmetric monoidal closed π-category $\mathscr{V}$ [2]. The $\mathscr{J}$-ary endofunctors are those that preserve $\mathbb{N}_{\mathscr{V}}$-flat colimits (incl. sifted colimits); the $\mathscr{J}$-ary monads correspond to the Borceux-Day enriched finite power theories of [2].

- Any eleutheric subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ in a locally bounded $\mathscr{V}$-category $\mathscr{C}$ over a locally bounded closed category $\mathscr{V}$.

Signatures

Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be a subcategory of arities. A $\mathscr{J}$*-signature* in $\mathscr{C}$ is just a functor $\Sigma : \mathbf{ob}\,\mathscr{J} \to \mathscr{C}_0$. We have a canonical forgetful functor $V : \mathbf{End}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}$.

Proposition

The forgetful functor $V : \mathbf{End}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}$ *is monadic.*

The free $\mathscr{J}$-ary endofunctor on a signature Σ is the *polynomial* endofunctor $H_\Sigma : \mathscr{C} \to \mathscr{C}$ given by

$$(X \in \mathbf{ob}\mathscr{C}) \qquad H_\Sigma X := \coprod_{J \in \mathscr{J}} \mathscr{C}(J, X) \otimes \Sigma J.$$

Free $\mathscr{J}$-ary Monads

- We also have a forgetful functor $W : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{End}_{\mathscr{J}}(\mathscr{C})$, which we also want to show has a left adjoint (and hence is monadic), and that $U := V \circ W : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}$ is monadic.
- Kelly and Power show in [5] that W has a left adjoint when $\mathscr{J} = \mathscr{C}_\alpha$ for a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally α-presentable $\mathscr{V}$, and Lack shows in [6] that U is monadic.
- Bourke and Garner show in [3] that if $\mathscr{C}$ is a locally presentable $\mathscr{V}$-category over a locally presentable $\mathscr{V}$, then $U : \mathbf{Mnd}_{\mathscr{J}\,\mathbf{Nerv}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}$ is monadic.

Algebraically Free $\mathscr{J}$-ary Monads

Theorem

If $H \in \mathbf{End}_{\mathscr{J}}(\mathscr{C})$, then $G : H\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint $F : \mathscr{C} \to H\text{-}\mathbf{Alg}$, and the $\mathscr{V}$-monad GF is an algebraically free $\mathscr{V}$-monad on H, i.e. $GF\text{-}\mathbf{Alg} \cong H\text{-}\mathbf{Alg}$, which is moreover $\mathscr{J}$-ary. So $W : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{End}_{\mathscr{J}}(\mathscr{C})$ has a left adjoint and is monadic.

If $\mathbb{T}_\Sigma$ is the free $\mathscr{J}$-ary monad on a signature Σ, then one can show that $\mathbb{T}_\Sigma\text{-}\mathbf{Alg} \cong H_\Sigma\text{-}\mathbf{Alg} \cong \Sigma\text{-}\mathbf{Alg}$, the $\mathscr{V}$-category of Σ-*algebras*.

Presentations of $\mathscr{J}$-ary Monads

- Let $j : \mathscr{J} \hookrightarrow \mathscr{C}$ be a subcategory of arities. A *$\mathscr{J}$-presentation* is a pair of $\mathscr{J}$-signature morphisms $\alpha, \beta : \Gamma \rightrightarrows U(\mathbb{T}_\Sigma)$, where Γ, Σ are $\mathscr{J}$-signatures and $\mathbb{T}_\Sigma$ is the free $\mathscr{J}$-ary monad on Σ. Γ should be thought of as the signature of 'equations'.

- We want to show that any $\mathscr{J}$-presentation P generates a $\mathscr{J}$-ary monad $\mathbb{T}_P$, i.e. that there is a coequalizer $\mathbb{T}_\Gamma \rightrightarrows \mathbb{T}_\Sigma \twoheadrightarrow \mathbb{T}_P$ in $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ which is preserved by the semantics functor $\mathbf{Alg} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to (\mathscr{V}\mathbf{CAT}/\mathscr{C})^{\mathbf{op}}$.

- If $\mathbb{M} : \mathcal{K} \to \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ is a small diagram, then we will write $G^{\mathbb{M}} : \mathbb{M}\text{-}\mathbf{Alg} \to \mathscr{C}$ for the limit of $\mathbf{Alg} \circ \mathbb{M}^{\mathbf{op}} : \mathcal{K}^{\mathbf{op}} \to \mathscr{V}\mathbf{CAT}/\mathscr{C}$.

Algebraic Colimits of $\mathscr{J}$-ary Monads

A colimit in $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ is *algebraic* if it is preserved by $\mathbf{Alg} : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to (\mathscr{V}\mathbf{CAT}/\mathscr{C})^{\mathbf{op}}$.

Theorem

If $\mathbb{M} : \mathcal{K} \to \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ is a small diagram, then $G^{\mathbb{M}} : \mathbb{M}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint $F^{\mathbb{M}} : \mathscr{C} \to \mathbb{M}\text{-}\mathbf{Alg}$, and the $\mathscr{V}$-monad $G^{\mathbb{M}}F^{\mathbb{M}}$ is an algebraic colimit of $\mathbb{M}$, which is moreover $\mathscr{J}$-ary. So $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ has small algebraic colimits. In particular, any $\mathscr{J}$-presentation generates a $\mathscr{J}$-ary monad.

If $\mathbb{T}_P$ is the $\mathscr{J}$-ary monad generated by a presentation P, then one can show that $\mathbb{T}_P\text{-}\mathbf{Alg} \cong P\text{-}\mathbf{Alg}$, the sub-$\mathscr{V}$-category of $\Sigma\text{-}\mathbf{Alg}$ consisting of the Σ-algebras that *satisfy* the presentation P.

Monadicity of $\mathscr{J}$-ary Monads

Using [6, Theorem 2], we have also shown:

Theorem

The forgetful functor $U : \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}) \to \mathbf{Sig}_{\mathscr{J}}$ *is monadic.*

In particular, it follows that any $\mathscr{J}$-ary monad on $\mathscr{C}$ has a $\mathscr{J}$-presentation.

Locally Bounded $\mathscr{V}$-Categories

Theorem

Let $\mathscr{C}$ be a locally bounded $\mathscr{V}$-category over a locally bounded closed category $\mathscr{V}$. Then any subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ is bounded.

Thus, all of our results hold for any eleutheric subcategory of arities in any locally bounded $\mathscr{V}$-category $\mathscr{C}$ over a locally bounded closed category $\mathscr{V}$.

Examples

We have the following examples (so far) of eleutheric subcategories of arities in locally bounded $\mathscr{V}$-categories:

- The subcategory of arities $j : \mathscr{C}_\alpha \hookrightarrow \mathscr{C}$ in a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$ over a locally α-presentable $\mathscr{V}$.
- The subcategory of arities $j : \mathscr{C}_{\mathbb{D}} \hookrightarrow \mathscr{C}$ in a locally $\mathbb{D}$-presentable $\mathscr{V}$-category over a locally $\mathbb{D}$-presentable closed category $\mathscr{V}$ (for a sound doctrine $\mathbb{D}$).

Examples

- The subcategory of arities $j : \{I\} \hookrightarrow \mathscr{V}$ in a (locally bounded) closed category $\mathscr{V}$.
- The subcategory of arities $j : \mathbb{N}_{\mathscr{V}} \hookrightarrow \mathscr{V}$ in a (locally bounded) symmetric monoidal closed π-category $\mathscr{V}$ [2].
- The subcategory of arities $j : \mathscr{C}_{\Phi} \hookrightarrow \mathscr{C}$ in a locally Φ-presentable $\mathscr{V}$-category over a locally bounded and $\mathscr{E}$-cowellpowered $\mathscr{V}$ (for locally small Φ satisfying Axiom A).

Conclusions

- We have developed a framework for studying signatures and presentations for enriched ($\mathscr{J}$-ary) monads that includes locally bounded enriched categories and arbitrary π-categories.
- Specifically, we have seen that free $\mathscr{J}$-ary monads and presentations for $\mathscr{J}$-ary monads can be obtained for any bounded and eleutheric subcategory of arities $j : \mathscr{J} \hookrightarrow \mathscr{C}$ in any cocomplete and cotensored $\mathscr{V}$-factegory $\mathscr{C}$.
- In particular, our results apply to any eleutheric subcategory of arities in any locally bounded $\mathscr{V}$-category over a locally bounded closed category $\mathscr{V}$.

Thank you!

E-mail: parkerj@brandonu.ca
Website: www.jasonparkermath.com

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