Enriched algebraic theories, monads, and varieties

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Introduction

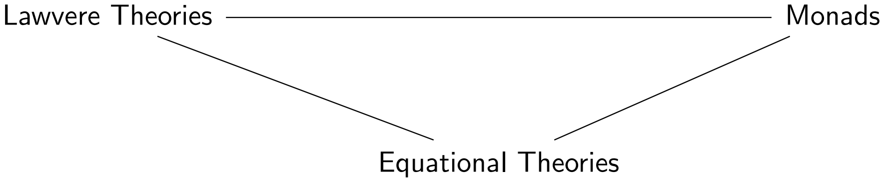
- My first postdoc was from September 2020–August 2023 at Brandon University in Brandon, Manitoba (even colder than Calgary), where I worked with Rory Lucyshyn-Wright. We mainly worked on enriched algebraic theories, enriched monads, and enriched varieties.

Rory:

Introduction

- I will first give a historical overview of the topic, and then describe the new contributions that Rory and I made.
- I will conclude by mentioning some of my recent, current, and future research on this topic.

The trinity of categorical algebra



Back to the beginning: Birkhoff, Lawvere, Linton

- Birkhoff [Bir35] originated the study of **universal algebra** in the 1930s, which was then given a categorical formulation by Lawvere [Law63] and Linton [Lin66] in the 1960s. Birkhoff defined a general notion of **(equational) variety of algebras**, which is a class of algebraic structures axiomatized by equations. E.g. the varieties of monoids, groups, commutative rings with unit, lattices, and many more.

- A **(finitary) signature** is a set Σ of **operation symbols** equipped with the assignment to each operation symbol $\sigma \in \Sigma$ of an **arity** $n \geq 0$. A Σ-**algebra** A is a set A equipped with a function $\sigma^A : A^n \to A$ for each $\sigma \in \Sigma$ of arity $n \geq 0$.

- A **morphism of** Σ-**algebras** $f : A \to B$ is a function $f : A \to B$ such that $f \circ \sigma^A = \sigma^B \circ f^n : A^n \to B$ for each $\sigma \in \Sigma$. We have the category Σ-**Alg** of Σ-algebras and a forgetful functor $U^\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathbf{Set}$.

Equational theories I

- Given a context of variables $\vec{v} \equiv v_1, \ldots, v_n$, one can recursively define the set $\mathbf{Term}(\Sigma; \vec{v})$ of **Σ-terms in context** $\vec{v}$ as follows: each v_i $(1 \leq i \leq n)$ is a Σ-term in context $\vec{v}$; and if $\sigma \in \Sigma$ has arity $m \geq 0$ and $t_1, \ldots, t_m$ are Σ-terms in context $\vec{v}$, then $\sigma(t_1, \ldots, t_m)$ is a Σ-term in context $\vec{v}$.

- Given a Σ-algebra A, each Σ-term t in context $\vec{v}$ induces an **interpretation function** $t^A : A^n \to A$.

- A **(syntactic) Σ-equation in context** $\vec{v}$ is an expression of the form $s \doteq t$ for Σ-terms s, t in context $\vec{v}$. A Σ-algebra A **satisfies** $s \doteq t$ if $s^A = t^A : A^n \to A$.

Equational theories II

- An **equational theory** is a pair $\mathcal{T} = (\Sigma, \mathcal{E})$ consisting of a signature Σ and a set $\mathcal{E}$ of syntactic Σ-equations in context. A $\mathcal{T}$-**algebra** is a Σ-algebra that satisfies each equation in $\mathcal{E}$. We have the full subcategory $\mathcal{T}\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$ and a forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to \mathbf{Set}$, so that $\mathcal{T}\text{-}\mathbf{Alg}$ can be regarded as an object of the slice category $\mathbf{CAT}/\mathbf{Set}$.

Varieties and finitary monads

- A **variety** is an object of $\mathbf{CAT}/\mathbf{Set}$ of the form $\mathcal{T}$-$\mathbf{Alg}$ for some equational theory $\mathcal{T}$. Examples include the varieties of sets, monoids, groups, commutative rings with unit, lattices, and many more.

- For each equational theory $\mathcal{T}$, the forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to \mathbf{Set}$ has a left adjoint $F^{\mathcal{T}} : \mathbf{Set} \to \mathcal{T}\text{-}\mathbf{Alg}$, and the resulting monad on $\mathbf{Set}$ is **finitary**, meaning that $U^{\mathcal{T}}F^{\mathcal{T}} : \mathbf{Set} \to \mathbf{Set}$ preserves filtered colimits. Conversely, for every finitary monad $\mathbb{T}$ on $\mathbf{Set}$, there is an equational theory $\mathcal{T}$ such that $\mathbb{T}\text{-}\mathbf{Alg} \cong \mathcal{T}\text{-}\mathbf{Alg}$ in $\mathbf{CAT}/\mathbf{Set}$. This correspondence extends to a dual equivalence

$$\mathbf{Var} \simeq \mathbf{Mnd}_f(\mathbf{Set})^{\mathbf{op}}.$$

Lawvere theories I

- Lawvere [Law63] discovered a purely categorical formulation of varieties in terms of **Lawvere theories**. A **Lawvere theory** is a category $\mathscr{T}$ with finite products equipped with an identity-on-objects functor $\tau : \mathbf{FinCard}^{\mathbf{op}} \to \mathscr{T}$ that preserves finite products. A $\mathscr{T}$-**algebra** is a functor $A : \mathscr{T} \to \mathbf{Set}$ that preserves finite products, and a morphism of $\mathscr{T}$-algebras is a natural transformation. We have a category $\mathscr{T}$-$\mathbf{Alg}$ and a forgetful functor $U^{\mathscr{T}} : \mathscr{T}\text{-}\mathbf{Alg} \to \mathbf{Set}$ given by $A \mapsto A(1)$.

- The functor $U^{\mathscr{T}}$ has a left adjoint, and the resulting monad on **Set** is finitary. Conversely, given a finitary monad $\mathbb{T}$ on **Set**, the full subcategory $\mathscr{T}$ of $\mathbb{T}$-**Alg** consisting of the free $\mathbb{T}$-algebras on finite cardinals is a Lawvere theory with $\mathbb{T}\text{-}\mathbf{Alg} \cong \mathscr{T}\text{-}\mathbf{Alg}$ in $\mathbf{CAT}/\mathbf{Set}$. This correspondence extends to an equivalence

$$\mathbf{Mnd}_f(\mathbf{Set}) \simeq \mathbf{Law}.$$

Lawvere theories II

- All told, in the classical setting of finitary universal algebra, we have the following (dual) equivalences between Lawvere theories, finitary monads on **Set**, and varieties, due to Birkhoff [Bir35], Lawvere [Law63] and Linton [Lin66]:

$$\mathbf{Law} \simeq \mathbf{Mnd}_f(\mathbf{Set}) \simeq \mathbf{Var}^{\mathbf{op}}.$$

Various settings for enriched algebra I

- After Lawvere and Linton, various researchers generalized the notions of Lawvere theory and finitary monad to the **enriched setting**, including:
  - [BD80] Francis Borceux and Brian Day, *Universal algebra in a closed category*, 1980.
  - [Pow99] John Power, *Enriched Lawvere theories*, 1999.
  - [NP09] Koki Nishizawa and John Power, *Lawvere theories enriched over a general base*, 2009.
  - [LR11] Stephen Lack and Jiří Rosický, *Notions of Lawvere theory*, 2011.
  - [LW16] Rory B. B. Lucyshyn-Wright, *Enriched algebraic theories and monads for a system of arities*, 2016.
  - [BG19] John Bourke and Richard Garner, *Monads and theories*, 2019.

Various settings for enriched algebra II

- However, virtually none of these frameworks developed a corresponding notion of **enriched variety**, and most of them were formulated in the **locally presentable** setting, which excludes many important (topological) categories in mathematics. Also, there was no framework for enriched algebra that captured all of these prior frameworks under one roof.

- So Rory and I set out to rectify all of these issues, which we did in the following papers and preprints:
  - [LWP22] *Presentations and algebraic colimits of enriched monads for a subcategory of arities*, TAC, 2022.
  - [LWP23a] *Diagrammatic presentations of enriched monads and varieties for a subcategory of arities*, ACS, 2023.
  - [LWP23b] *Enriched structure-semantics adjunctions and monad-theory equivalences for subcategories of arities*, Preprint, 2023.
  - [LWP23c] *Diagrammatic presentations of enriched monads and the axiomatics of enriched algebra*, In preparation, 2023.

Moving towards the enriched setting...

- We can refer to the full subcategory $j : \mathbf{FinCard} \hookrightarrow \mathbf{Set}$ as a **subcategory of arities** (because it is **dense**). A contravariant functor $\mathbf{FinCard}^{\mathbf{op}} \to \mathbf{Set}$ is a j-**nerve** if it is of the form $\mathbf{Set}(j-, X)$ for some set X.

- A Lawvere theory can then be equivalently defined as a category $\mathscr{T}$ equipped with an identity-on-objects functor $\tau : \mathbf{FinCard}^{\mathbf{op}} \to \mathscr{T}$ such that for each $n \in \mathbb{N}$, the functor $\mathscr{T}(n, \tau-) : \mathbf{FinCard}^{\mathbf{op}} \to \mathbf{Set}$ is a j-nerve.

- A $\mathscr{T}$-algebra can also be equivalently defined as a functor $A : \mathscr{T} \to \mathbf{Set}$ such that $A \circ \tau : \mathbf{FinCard}^{\mathbf{op}} \to \mathbf{Set}$ is a j-nerve.

Subcategories of arities

- Let $\mathscr{V}$ be a complete and cocomplete symmetric monoidal closed category (a **cosmos**), and let $\mathscr{C}$ be a complete and cocomplete $\mathscr{V}$-category. A **subcategory of arities** is a dense full sub-$\mathscr{V}$-category $j : \mathscr{J} \hookrightarrow \mathscr{C}$.

- E.g. with $\mathscr{V} = \textbf{Set}$ and $\mathscr{C} = \textbf{Set}$, we have the subcategory of arities $j : \textbf{FinCard} \hookrightarrow \textbf{Set}$.

$\mathscr{J}$-theories

- A j-**nerve** is a $\mathscr{V}$-functor $\mathscr{J}^{\mathbf{op}} \to \mathscr{V}$ of the form $\mathscr{C}(j-, C)$ for some object C of $\mathscr{C}$.

- A $\mathscr{J}$-**theory** is a $\mathscr{V}$-category $\mathscr{T}$ equipped with an identity-on-objects $\mathscr{V}$-functor $\tau : \mathscr{J}^{\mathbf{op}} \to \mathscr{T}$ such that for each $J \in \mathbf{ob}\,\mathscr{J}$, the $\mathscr{V}$-functor $\mathscr{T}(J, \tau-) : \mathscr{J}^{\mathbf{op}} \to \mathscr{V}$ is a j-nerve.

- A $\mathscr{T}$-**algebra** is a $\mathscr{V}$-functor $A : \mathscr{T} \to \mathscr{V}$ such that $A \circ \tau : \mathscr{J}^{\mathbf{op}} \to \mathscr{V}$ is a j-nerve. There is a $\mathscr{V}$-category $\mathscr{T}$-**Alg** and a forgetful $\mathscr{V}$-functor $U^{\mathscr{T}} : \mathscr{T}\text{-}\mathbf{Alg} \to \mathscr{C}$. With $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C} = \mathbf{Set}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Set}$, we recover Lawvere theories and their (categories of) algebras.

The enriched monad–theory equivalence I

- A $\mathscr{V}$-monad $\mathbb{T}$ on $\mathscr{C}$ is $\mathscr{J}$**-ary** (or $\mathscr{J}$**-nervous**) if it satisfies some technical conditions [LWP23b]. With $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C} = \mathbf{Set}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Set}$, we recover finitary monads on **Set**.

Theorem ([LWP23b])

Suppose that the subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ is **amenable**. *Then there is an equivalence*

$$\mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$$

between the category of $\mathscr{J}$-theories and the category of $\mathscr{J}$-ary $\mathscr{V}$-monads on $\mathscr{C}$, which respects semantics in an appropriate sense.

- With $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C} = \mathbf{Set}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Set}$, we recover the classical equivalence between Lawvere theories and finitary monads on **Set**.

The enriched monad–theory equivalence II

- Moreover, this Theorem recovers the enriched monad–theory equivalences established in all of the aforementioned papers (and more): [BD80], [Pow99], [NP09], [LR11], [LW16], [BG19]. So we have united all of these frameworks under one roof!

- Examples of **amenable** subcategories of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ include:

  - Any **eleutheric** subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ (e.g. **FinCard** $\hookrightarrow$ **Set**; $\mathscr{C}_\alpha \hookrightarrow \mathscr{C}$ for a locally α-presentable $\mathscr{V}$-category $\mathscr{C}$; **FinCard** $\hookrightarrow \mathscr{V}$ for a complete and cocomplete cartesian closed $\mathscr{V}$).

  - Any small subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$**-sketchable** $\mathscr{V}$**-category** $\mathscr{C}$ enriched over a **locally bounded** $\mathscr{V}$. In particular, any small subcategory of arities in a locally bounded $\mathscr{V}$, e.g. in any topological category over **Set**.

The enriched monad–theory equivalence III

- The second class of examples is completely new: none of the aforementioned papers [BD80], [Pow99], [NP09], [LR11], [LW16], [BG19] had managed to incorporate topological bases of enrichment in any significant way, because they almost always used **locally presentable** bases of enrichment.

What about enriched varieties? I

- With $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C} = \mathbf{Set}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Set}$, there is an established notion of variety. None of the prior papers developed an enriched notion of variety, so Rory and I did that in [LWP22, LWP23a, LWP23c].

- Well, actually, the following papers went partway towards defining enriched notions of variety in certain restricted settings:

  - [KP93] G. M. Kelly and A. J. Power, *Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads*, 1993.

  - [KL93] G. M. Kelly and Stephen Lack, *Finite-product-preserving functors, Kan extensions and strongly-finitary* 2*-monads*, 1993.

  But their notions were quite far from the classical notions, and not as concrete and user-friendly as the ones that we ultimately developed.

What about enriched varieties? II

- To motivate this enriched notion of variety, let's start by reformulating the classical notion of variety in a way that will more readily admit generalization to the enriched setting.

- First, note that a signature Σ can be equivalently formulated as a family of sets $(\Sigma n)_{n \in \mathbf{FinCard}}$. A Σ-algebra A can then be (re)formulated as a set A equipped with, for each $n \in \mathbb{N}$, an **interpretation function**

$$\mathfrak{i}_n^A : \Sigma(n) \to \mathbf{Set}\left(A^n, A\right) = \mathbf{Set}\left(\mathbf{Set}(n, A), A\right).$$

What about enriched varieties? III

- Now let's look at Σ-terms. It is a classical fact/result that the Σ-terms t in context $\vec{v} \equiv v_1, \ldots, v_n$ correspond bijectively to **algebraic Σ-operations of arity** n, i.e. natural transformations

$$\omega^t : \mathbf{Set}\left(n, U^\Sigma -\right) \Longrightarrow U^\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathbf{Set},$$

  in such a way that

$$t^A = \omega^t_A : A^n = \mathbf{Set}\,(n, A) \to A$$

  for each Σ-algebra A.

- A syntactic Σ-equation $s \doteq t$ in context $\vec{v}$ can then be equivalently formulated as an **algebraic Σ-equation of arity** n, i.e. a pair $\omega^s \doteq \omega^t$ of algebraic Σ-operations of arity n. A Σ-algebra A satisfies $s \doteq t$ iff $\omega^s_A = \omega^t_A : A^n \to A$.

What about enriched varieties? IV

- In summary: an equational theory can be equivalently formulated as a pair $\mathcal{T} = (\Sigma, \mathcal{E})$ consisting of a **FinCard**-indexed family of sets $\Sigma = (\Sigma n)_{n \in \mathbf{FinCard}}$ and a set $\mathcal{E}$ of algebraic Σ-equations.
- This will be our basis for defining notions of enriched equational theory and enriched variety.

Equational $\mathscr{J}$-theories and $\mathscr{J}$-ary varieties I

- A $\mathscr{J}$-**signature** is an $\mathbf{ob}\,\mathscr{J}$-indexed family $\Sigma = (\Sigma J)_{J \in \mathbf{ob}\,\mathscr{J}}$ of objects of $\mathscr{C}$.

- A Σ-**algebra** is an object A of $\mathscr{C}$ equipped with, for each $J \in \mathbf{ob}\,\mathscr{J}$, an **interpretation** $\mathscr{C}$**-morphism**

$$\mathfrak{i}_J^A : \Sigma J \to [\mathscr{C}(J, A), A].$$

  We have a $\mathscr{V}$-category Σ-$\mathbf{Alg}$ of Σ-algebras and a forgetful $\mathscr{V}$-functor $U^\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$.

- An **algebraic** Σ-**operation** is a $\mathscr{V}$-natural transformation

$$\omega : \mathscr{C}\left(J, U^\Sigma -\right) \otimes P \Longrightarrow U^\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$$

  for a specified **arity** $J \in \mathbf{ob}\,\mathscr{J}$ and **parameter (object)** $P \in \mathbf{ob}\mathscr{C}$.

Equational $\mathscr{J}$-theories and $\mathscr{J}$-ary varieties II

- An **algebraic Σ-equation** is a pair $\omega \doteq \nu$ of algebraic Σ-operations with the same arity and parameter. A Σ-algebra A **satisfies** $\omega \doteq \nu$ when $\omega_A = \nu_A : \mathscr{C}(J, A) \otimes P \to A$.

- An **equational $\mathscr{J}$-theory** is a pair $\mathcal{T} = (\Sigma, \mathcal{E})$ consisting of a $\mathscr{J}$-signature Σ and a set $\mathcal{E}$ of algebraic Σ-equations. A $\mathcal{T}$**-algebra** is a Σ-algebra that satisfies each algebraic Σ-equation in $\mathcal{E}$. We have the full sub-$\mathscr{V}$-category $\mathcal{T}$-$\mathbf{Alg} \hookrightarrow \Sigma$-$\mathbf{Alg}$ and a forgetful $\mathscr{V}$-functor $U^{\mathcal{T}} : \mathcal{T}$-$\mathbf{Alg} \to \mathscr{C}$, so that $\mathcal{T}$-$\mathbf{Alg}$ may be regarded as an object of the slice category $\mathscr{V}$-$\mathbf{CAT}/\mathscr{C}$.

- A **$\mathscr{J}$-ary variety** is an object of $\mathscr{V}$-$\mathbf{CAT}/\mathscr{C}$ of the form $\mathcal{T}$-$\mathbf{Alg}$ for some equational $\mathscr{J}$-theory $\mathcal{T}$. We write $\mathbf{Var}_{\mathscr{J}}(\mathscr{C})$ for the category of $\mathscr{J}$-ary varieties (a full subcategory of $\mathscr{V}$-$\mathbf{CAT}/\mathscr{C}$).

Equational $\mathscr{J}$-theories and $\mathscr{J}$-ary varieties III

Theorem ([LWP23c])

Suppose that the subcategory of arities $\mathscr{J} \hookrightarrow \mathscr{C}$ is **strongly amenable**. *Then the category* $\mathbf{Var}_{\mathscr{J}}(\mathscr{C})$ *of $\mathscr{J}$-ary varieties is dually equivalent to the categories* $\mathbf{Th}_{\mathscr{J}}(\mathscr{C})$ *of $\mathscr{J}$-theories and* $\mathbf{Mnd}_{\mathscr{J}}(\mathscr{C})$ *of $\mathscr{J}$-ary $\mathscr{V}$-monads on $\mathscr{C}$:*

$$\mathbf{Var}_{\mathscr{J}}(\mathscr{C})^{\mathsf{op}} \simeq \mathbf{Th}_{\mathscr{J}}(\mathscr{C}) \simeq \mathbf{Mnd}_{\mathscr{J}}(\mathscr{C}).$$

These (dual) equivalences respect semantics. In particular, every equational $\mathscr{J}$-theory **presents** *a $\mathscr{J}$-theory and a $\mathscr{J}$-ary $\mathscr{V}$-monad on $\mathscr{C}$, while every $\mathscr{J}$-theory and every $\mathscr{J}$-ary $\mathscr{V}$-monad is* **presented by** *an equational $\mathscr{J}$-theory.*

With $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C} = \mathbf{Set}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Set}$, we recover the classical (dual) equivalences between varieties, Lawvere theories, and finitary monads on **Set**.

Examples of $\mathscr{J}$-ary varieties I

- With $\mathscr{C} = \mathscr{V}$ cartesian closed and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathscr{V}$: the $\mathscr{V}$-category of **internal** R**-modules** for an **internal ri(n)g** R in $\mathscr{V}$.
- With $\mathscr{V} = \mathbf{Set}$ and $\mathscr{C} = \mathbf{Grph}$ and $\mathscr{J} = \mathbf{Grph_{fp}} \hookrightarrow \mathbf{Grph}$: the category **Cat** of small categories.
- With $\mathscr{C} = \mathscr{V} = \mathbf{Cat}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Cat}$: the 2-category of small monoidal categories and strict monoidal functors.
- With $\mathscr{V}$ locally bounded and $\mathscr{C} = \mathbf{Grph}(\mathscr{V})$ (**internal graphs** in $\mathscr{V}$): the $\mathscr{V}$-category of **internal categories** in $\mathscr{V}$.

Examples of $\mathscr{J}$-ary varieties II

- With $\mathscr{C} = \mathscr{V} = \mathbf{CGTop}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{CGTop}$: the **CGTop**-category of **H-monoids**, i.e. internal 'monoids' in **CGTop** whose multiplication is only associative and unital up to specified homotopies.

- The **ordered equational theories** of Adámek–Dostál–Velebil [ADV22], with $\mathscr{V} = \mathbf{Pos}$ and $\mathscr{C} = \mathbf{Pos}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Pos}$.

- The **quantitative equational theories** of Mardare–Panangaden–Plotkin [MPP16], with $\mathscr{V} = \mathbf{Met}$ and $\mathscr{C} = \mathbf{Met}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \mathbf{Met}$.

Examples of $\mathscr{J}$-ary varieties III

- The **relational algebraic theories** of Ford–Milius–Schröder [FMS21], with $\mathscr{V} = \mathbb{T}$-**Mod** and $\mathscr{C} = \mathbb{T}$-**Mod** for a relational Horn theory $\mathbb{T}$ and $\mathscr{J} = \mathbb{T}\text{-}\mathbf{Mod}_{\mathbf{fp}} \hookrightarrow \mathbb{T}\text{-}\mathbf{Mod}$.

- The **continuous varieties** of Adámek–Dostál–Velebil [ADV23], with $\mathscr{V} = \omega\text{-}\mathbf{CPO}/\mathbf{DCPO}$ and $\mathscr{C} = \omega\text{-}\mathbf{CPO}/\mathbf{DCPO}$ and $\mathscr{J} = \mathbf{FinCard} \hookrightarrow \omega\text{-}\mathbf{CPO}/\mathbf{DCPO}$.

Current and future research I

- I have recently been focusing on specific settings where equational $\mathscr{J}$-theories and $\mathscr{J}$-ary varieties have an even more concrete and syntactic formulation. In particular, I have recently worked in the setting where $\mathscr{V}$ is topological over **Set**:
  - [Par23] Jason Parker, *Free algebras of topologically enriched multi-sorted equational theories*, Preprint, 2023.
- Currently I am working in the setting where $\mathscr{V} = \mathbb{T}$-**Mod** for a relational Horn theory $\mathbb{T}$, where equational $\mathscr{J}$-theories have an even more concrete formulation than in [Par23]. Examples of such $\mathscr{V}$ include **Pos**, **Met**, **Simp**, **ProbMet**, $\mathcal{Q}$-**Cat** for a commutative unital quantale $\mathcal{Q}$, and many more.

Current and future research II

- In these settings, I have been focusing on the case of **multi-sorted** equational $\mathscr{J}$-theories and $\mathscr{J}$-ary varieties, where one takes $\mathscr{C} = \mathscr{V}^{\mathcal{S}}$ and $\mathscr{J} = \mathbf{FinCard}_{\mathcal{S}} \hookrightarrow \mathscr{V}^{\mathcal{S}}$ for a set of sorts $\mathcal{S}$.

- Classical multi-sorted equational theories (with $\mathscr{V} = \mathbf{Set}$) have had many applications in algebraic specification, algebraic datatypes, and the (categorical) theory of databases, so I am hoping that enriched multi-sorted equational theories may have similar applications!

- I will talk about these further topics (and others) at future seminars!

Thank you!

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