Isotropy of Algebraic Theories

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Motivation and Basic Definitions

Given a category $\mathbb{C}$, the assignment

$$C \mapsto Aut(C)$$

is in general not functorial.

Given $f : D \to C$ in $\mathbb{C}$, then unless f is iso, there is no canonical group homomorphism

$$Aut(C) \to Aut(D).$$

We can introduce the isotropy group (functor)

$$\mathcal{Z} = \mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to Group$$

of $\mathbb{C}$ to solve this 'problem'.

Motivation and Basic Definitions

Given $C \in \mathbb{C}$, we set

$$\mathcal{Z}(C) = Aut(\mathbb{C}/C \to \mathbb{C}),$$

the group of natural automorphisms of the forgetful functor $\mathbb{C}/C \to \mathbb{C}$.

This assignment *is* functorial in C. An element $\alpha \in \mathcal{Z}(C)$ is then given by the following data:

1. An automorphism
$$\alpha_{id_C} : C \xrightarrow{\sim} C.$$
2. $\forall$ other morphism $f : D \to C$ in $\mathbb{C}$, an automorphism
$$\alpha_f : D \xrightarrow{\sim} D$$

   such that for any commuting triangle

Motivation and Basic Definitions

$$\begin{array}{ccc} D & \xrightarrow{h} & D' \\ & \searrow^{f_1} & \downarrow f_2 \\ & & C \end{array}$$

in $\mathbb{C}$, the following diagram commutes:

$$\begin{array}{ccc} D & \xrightarrow{\alpha_{f_1}} & D \\ h \downarrow & & \downarrow h \\ D' & \xrightarrow[\alpha_{f_2}]{} & D' \end{array}$$

An element of isotropy $\alpha \in \mathcal{Z}(C)$ could therefore also be referred to as a *coherent system of automorphisms* of C.

Isotropy Groups of Toposes

If $\mathcal{E}$ is a Grothendieck topos, then it can be shown that the isotropy group functor $\mathcal{Z}_{\mathcal{E}} : \mathcal{E}^{op} \to Group$ is representable by an internal group object $Z_{\mathcal{E}} \in \mathcal{E}$, called the *isotropy group* of $\mathcal{E}$.

The isotropy group of a Grothendieck topos $\mathcal{E}$ admits several different descriptions:

- Freyd introduced the notion of the *core* of a category which (informally speaking), if it exists, is a monoid in the category that represents the polymorphic unary operations present in the category. It can then be shown that the core of a Grothendieck topos $\mathcal{E}$ does exist, and that the isotropy group of $\mathcal{E}$ is then the group of invertible elements of the core. So elements of the isotropy group can be interpreted as polymorphic automorphisms in the topos.

Isotropy Groups of Toposes

- For any Grothendieck topos $\mathcal{E}$, there is a geometric theory $\mathbb{T}$ such that $\mathcal{E}$ is the classifying topos $\mathcal{B}(\mathbb{T})$ of $\mathbb{T}$. Then it can be shown that the isotropy group of $\mathcal{E} = \mathcal{B}(\mathbb{T})$ is the automorphism group of the universal $\mathbb{T}$-model $U_{\mathbb{T}} \in \mathcal{B}(\mathbb{T})$.
- Spencer Breiner has also shown that if we represent $\mathcal{E} = \mathcal{B}(\mathbb{T})$ as the topos of sheaves on the (topological) groupoid of $\mathbb{T}$-models, then the isotropy group of $\mathcal{E}$ is the sheaf of groups whose stalk at a $\mathbb{T}$-model M is the group of *definable* automorphisms of M.
- When $\mathcal{E} = Set^{\mathbb{C}^{op}}$ for a category $\mathbb{C}$, then the isotropy group $Z_{\mathcal{E}} \in Set^{\mathbb{C}^{op}}$ is exactly the isotropy functor

$$Z_{\mathcal{E}} = \mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to Group \hookrightarrow Set.$$

Isotropy Groups of Algebraic Theories

Let $\mathbb{T}$ be an algebraic/equational theory, i.e. a theory whose underlying language consists of a single sort, countably many variables of this sort, and function symbols of potentially all finite arities. The (non-logical) axioms of $\mathbb{T}$ are equations between terms over this language.

Let $fp\mathbb{T}mod$ be the category of all finitely presented $\mathbb{T}$-models and $\mathbb{T}$-model homomorphisms. It is well known that $\mathbb{T}$ has a classifying topos, which is the category $Set^{fp\mathbb{T}mod}$ of all covariant functors from $fp\mathbb{T}mod$ to Set. So for any Grothendieck topos $\mathcal{E}$, there is an equivalence of categories

$$Geom(\mathcal{E}, Set^{fp\mathbb{T}mod}) \simeq \mathbb{T}\text{-}mod(\mathcal{E}),$$

natural in $\mathcal{E}$.

Isotropy Groups of Algebraic Theories

We will now analyze the isotropy group of this classifying topos $Set^{fp\mathbb{T}mod}$, which we refer to as the isotropy group of the algebraic theory $\mathbb{T}$.

This will provide a new invariant of algebraic theories (in the sense that equivalent theories will have the 'same' isotropy groups).

Recall that the isotropy group of $Set^{fp\mathbb{T}mod}$ is the isotropy functor

$$\mathcal{Z}_{\mathbb{T}} : fp\mathbb{T}mod \to Group \hookrightarrow Set.$$

Given $M \in fp\mathbb{T}mod$, we refer to $\mathcal{Z}_{\mathbb{T}}(M)$ as the isotropy group of the $\mathbb{T}$-model M, where

$$\mathcal{Z}_{\mathbb{T}}(M) = Aut(M/fp\mathbb{T}mod \to fp\mathbb{T}mod).$$

Bergman's Theorem

As the starting point for our analysis, we state a theorem proved by George Bergman. First, given any group G and element $s \in G$, we can define a corresponding element $\hat{s} \in Aut(G/Group \to Group)$ as follows. For any group morphism $f : G \to H$, we set

$$\hat{s}_f : H \xrightarrow{\sim} H$$

$$h \mapsto f(s)hf(s)^{-1}$$

$\forall h \in H$ (so $\hat{s}_f$ is the inner automorphism of H induced by $f(s) \in H$).

Then we obtain a group homomorphism

$$\hat{} : G \to Aut(G/Group \to Group).$$

Bergman's Theorem

Then Bergman proved the following theorem.

Theorem (Bergman)

The map

$$\hat{} : G \to Aut(G/Group \to Group)$$

is in fact a group **isomorphism**.

It follows that a group automorphism $f : G \xrightarrow{\sim} G$ *is* **inner** *iff there is a coherent system of automorphisms* α *of* G *such that*

$$\alpha_{id_G} = f.$$

Bergman's Theorem

This result remains true if we replace groups by finitely presented groups. It can then be used to fully characterize the isotropy group of the algebraic theory of groups in terms of conjugation.

In what follows, we will see that the isotropy group of a general algebraic theory can be thought of as specifying a notion of *formal conjugation* or *inner automorphism* for (models of) that theory.

Syntactic Characterizations

Recall that for any algebraic theory $\mathbb{T}$ and $M \in \mathit{fp}\mathbb{T}\mathit{mod}$, the isotropy group $\mathcal{Z}_{\mathbb{T}}(M)$ is the group $\mathit{Aut}(M/\mathit{fp}\mathbb{T}\mathit{mod} \to \mathit{fp}\mathbb{T}\mathit{mod})$ of all coherent systems of automorphisms of M (in $\mathit{fp}\mathbb{T}\mathit{mod}$).

This is a purely categorical description of $\mathcal{Z}_{\mathbb{T}}(M)$. We will now show how to provide a 'semi'-syntactic description, and then a fully syntactic description.

Syntactic Characterizations

The proof of Bergman's theorem involved some techniques that inspired our semi-syntactic characterization of the isotropy group of an algebraic theory $\mathbb{T}$.

First, if $M \in fp\mathbb{T}mod$, then $M\langle x_1, \ldots, x_n\rangle$ is the (finitely presented) $\mathbb{T}$-model gained by adjoining indeterminates $x_1, \ldots, x_n$ to M. It is the coproduct of M with the free $\mathbb{T}$-model on the n generators $x_1, \ldots, x_n$.

Then an element $[t(x_1, \ldots, x_n)] \in M\langle x_1, \ldots, x_n\rangle$ is (the congruence class of) a word in $x_1, \ldots, x_n$, the elements of M, and the function symbols of $\mathbb{T}$, modulo the equations of $\mathbb{T}$ and relations among elements of M.

Syntactic Characterizations

If $[t(x_1, \ldots, x_n)]$ is any element of $M\langle x_1, \ldots, x_n\rangle$, then $[t]$ induces a function

$$[t]^M : M^n \to M,$$

given by substitution into the indeterminates $x_1, \ldots, x_n$ and then evaluation in M.

And if $h : M \to N$ is any morphism of $fp\mathbb{T}mod$, then h induces a unique map

$$h_{x_1,\ldots,x_n} : M\langle x_1, \ldots, x_n\rangle \to N\langle x_1, \ldots, x_n\rangle$$

in the obvious way.

We can also view $M\langle x\rangle$ as a monoid, under the operation of substitution and with $[x]$ as the identity element. So $[t] \cdot [s] = [t[s/x]]$. Then the above process (where $n = 1$) yields a monoid homomorphism

$$M\langle x\rangle \to End(M),$$

where $End(M)$ is the monoid of endofunctions of M.

First Syntactic Characterization

Now we make the following definition:

Definition

Let $\mathbb{T}$ be an algebraic theory. For any model $M \in fp\mathbb{T}mod$, let $\mathbf{M^{coh}}\langle \mathbf{x} \rangle$ be the set of all $[t(x)] \in M\langle x \rangle$ such that for any morphism $h : M \to N$ in $fp\mathbb{T}mod$, the induced function

$$h_x([t])^N : N \to N$$

is a $\mathbb{T}$-automorphism of N.

First Syntactic Characterization

Then $M^{coh}\langle x \rangle$ is a group (as a submonoid of $M\langle x \rangle$), and we have

Theorem (Hofstra, Parker, Scott)

Let $\mathbb{T}$ be an algebraic theory with isotropy group $\mathcal{Z}_{\mathbb{T}} \in Set^{fp\mathbb{T}mod}$. Then $\forall M \in fp\mathbb{T}mod$ there is a group isomorphism

$$\mathcal{Z}_{\mathbb{T}}(M) \cong M^{coh}\langle x \rangle,$$

natural in M.

So the isotropy group (functor) of $\mathbb{T}$ sends every $M \in fp\mathbb{T}mod$ to the group of all elements $[t(x)]$ in $M\langle x \rangle$ whose homomorphic images always induce $\mathbb{T}$-automorphisms.

Second Syntactic Characterization

We now want to determine if there is a (purely syntactic) condition on elements $[t(x)] \in M\langle x \rangle$ that will characterize the sets $M^{coh}\langle x \rangle$ without quantifying over arbitrary homomorphisms out of M. It turns out that there is such a condition, which we describe as follows.

First, note that an element $[t(x)]$ in the monoid $M\langle x \rangle$ is invertible (by definition) iff there is an element $[s(x)] \in M\langle x \rangle$ such that

$$[t[s/x]] = [x] = [s[t/x]].$$

Second Syntactic Characterization

Definition

Let $\mathbb{T}$ be an algebraic theory with $M \in fp\mathbb{T}mod$. Let $M^{hom}\langle x \rangle$ be the set of all $[t(x)] \in M\langle x \rangle$ such that:

1. $[t(x)]$ is invertible in the monoid $M\langle x \rangle$.
2. For any n-ary function symbol f of $\mathbb{T}$ ($n \geq 0$) and indeterminates $x_1, \ldots, x_n$, the equality

$$[f(t(x_1), \ldots, t(x_n))] = [t[f(x_1, \ldots, x_n)/x]]$$

   holds in the $\mathbb{T}$-model $M\langle x_1, \ldots, x_n \rangle$. We say that $[t(x)]$ **commutes generically** with all operations of $\mathbb{T}$.

It is easy to prove that if $[t(x)] \in M^{hom}\langle x \rangle$, then $t(x)$ has at least one occurrence of x (assuming that $\mathbb{T}$ does not prove $x = y$ for distinct variables x, y).

Second Syntactic Characterization

Now we have the second, fully syntactic characterization of the isotropy group of an algebraic theory $\mathbb{T}$:

Theorem (Hofstra, Parker, Scott)

Let $\mathbb{T}$ be an algebraic theory with $M \in fp\mathbb{T}mod$. Then

$$M^{coh}\langle x \rangle = M^{hom}\langle x \rangle.$$

So if $\mathcal{Z}_{\mathbb{T}} \in Set^{fp\mathbb{T}mod}$ is the isotropy group of $\mathbb{T}$, then $\forall M \in fp\mathbb{T}mod$ we have

$$\mathcal{Z}_{\mathbb{T}}(M) \cong M^{hom}\langle x \rangle,$$

natural in M.

So the isotropy group (functor) of $\mathbb{T}$ sends every $M \in fp\mathbb{T}mod$ to the group of all elements $[t(x)] \in M\langle x \rangle$ that are (substitutionally) invertible and commute generically with all operations of $\mathbb{T}$.

Examples

- If $\mathbb{T}$ has no axioms, then the isotropy group of $\mathbb{T}$ is trivial, i.e. $\forall M \in fp\mathbb{T}mod$ we have $\mathcal{Z}_{\mathbb{T}}(M) = \{[x]\} \cong 1$, the trivial group.
- If $\mathbb{T}$ is the theory of groups, then Bergman essentially proved that $\forall G \in fpGroup$ we have

$$\mathcal{Z}_{\mathbb{T}}(G) = \{[gxg^{-1}] \in G\langle x\rangle \mid g \in G\} \cong G.$$

- If $\mathbb{T}$ is the theory of monoids, then $\forall M \in fpMonoid$ we have

$$\mathcal{Z}_{\mathbb{T}}(M) = \{[mxm'] \in M\langle x\rangle \mid m \text{ is invertible in } M \text{ and } m' = m^{-1}\}.$$

Examples

- If $\mathbb{T}$ is the theory of abelian groups, then $\forall G \in fpAb$ we have
$$\mathcal{Z}_{\mathbb{T}}(G) = \{[x], [-x]\} \cong \mathbb{Z}_2.$$
- If $\mathbb{T}$ is the theory of commutative monoids, then the isotropy group of $\mathbb{T}$ is trivial.
- If $\mathbb{T}$ is the theory of non-commutative rings with 1, then $\forall R \in fpRing$ we have
$$\mathcal{Z}_{\mathbb{T}}(R) = \{[rxr^{-1}] \in R\langle x \rangle \mid \ r \text{ is a unit } \}.$$
- If $\mathbb{T}$ is the theory of commutative rings with 1, then the isotropy group of $\mathbb{T}$ is trivial.
- If $\mathbb{T}$ is the theory of R-modules for some commutative ring R, then $\forall M \in fpRmod$, we have that
$$\mathcal{Z}_{\mathbb{T}}(M) = \{(0_M, rx) \in M \oplus \langle x \rangle \mid \ r \text{ is a unit } \}.$$

Examples

- Let $\mathbb{T}$ be the theory of a bijection, so that $\mathbb{T}$ has two unary operation symbols f and f^{-1} with the axioms

  $$f(f^{-1}(x)) = x = f^{-1}(f(x)).$$

  Then $\forall M \in fp\mathbb{T}mod$, we have that

  $$\mathcal{Z}_{\mathbb{T}}(M) = \{[f^n(x)] \in M\langle x\rangle \mid n \in \mathbb{Z}\} \cong \mathbb{Z}.$$

- If $\mathbb{T}$ is the theory of (not necessarily bounded or distributive) lattices over the signature $\{\wedge, \vee\}$, then the isotropy group of $\mathbb{T}$ is trivial. The proof of this relies on Whitman's solution of the word problem for finitely presented lattices.

Examples

- The theories of racks and quandles axiomatize the notion of conjugation (without reference to multiplication or inverses). Both theories are expressed over a signature with two binary function symbols $\triangleleft, \triangleleft^{-1}$. The axioms for the theory of racks are as follows:
  - $x \triangleleft (y \triangleleft z) = (x \triangleleft y) \triangleleft (x \triangleleft z)$ (and similarly for $\triangleleft^{-1}$).
  - $(x \triangleleft y) \triangleleft^{-1} y = x$ (and dually, switching $\triangleleft$ and $\triangleleft^{-1}$).

  The axioms for the theory of quandles are the axioms for the theory of racks, together with the following axioms:

$$x \triangleleft x = x = x \triangleleft^{-1} x.$$

Examples

If $\mathcal{Q}_n$ is the free quandle on n generators and $\mathcal{F}_n$ is the free group on n generators, then we proved that

$$\mathcal{Z}(\mathcal{Q}_n) \cong \mathcal{F}_n.$$

If $\mathcal{R}_n$ is the free rack on n generators, then we proved that

$$\mathcal{Z}(\mathcal{R}_n) \cong \mathbb{Z} \times \mathcal{F}_n.$$

The proofs of these results relied on the translation of the word problems for free racks and quandles into the word problem for free groups, given by Dehornoy.

Conclusion

Many of these examples suggest that the isotropy group of $\mathbb{T}$ has a close connection to the "inner automorphisms" of $\mathbb{T}$-models.

Indeed, they suggest that for a general algebraic theory $\mathbb{T}$, an automorphism $f \in Aut(M)$ of a $\mathbb{T}$-model M should be called *inner* if there is an element of isotropy

$$\alpha \in \mathcal{Z}_{\mathbb{T}}(M) = Aut(M/fp\mathbb{T}mod \to fp\mathbb{T}mod)$$

such that

$$\alpha_{id_M} = f,$$

equivalently if there is an element $[t(x)] \in M^{hom}\langle x \rangle \cong \mathcal{Z}_{\mathbb{T}}(M)$ such that

$$f = [t]^M : M \to M.$$

Thank you!