

Presentations and algebraic colimits of enriched monads

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Introduction

- Signatures and presentations for monads and theories have been previously studied mainly in the context of *locally presentable* enriched categories; see e.g.
 - ▶ [4] G.M. Kelly and A.J. Power. Adjunctions whose counits are coequalizers, and presentations of finitary enriched monads. *Journal of Pure and Applied Algebra* Vol. 89 (1993) 163-179.
 - ▶ [5] S. Lack. On the monadicity of finitary monads. *Journal of Pure and Applied Algebra* Vol. 140 (1999) 65-73.
 - ▶ [2] J. Bourke and R. Garner. Monads and theories. *Advances in Mathematics* Vol. 351 (2019) 1024-1071.
- In this talk, we discuss recent work on developing a framework for such phenomena that subsumes a much wider class of enriched categories, including *locally bounded* enriched categories and the symmetric monoidal closed π -categories of [1].

Subcategories of Arities

- A subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a \mathcal{V} -category \mathcal{C} is a small, full, and dense sub- \mathcal{V} -category.
- A subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is *eleutheric* [7] if every \mathcal{V} -functor $F : \mathcal{J} \rightarrow \mathcal{C}$ has a left Kan extension along j , which is moreover preserved by $\mathcal{C}(J, -) : \mathcal{C} \rightarrow \mathcal{V}$ for each $J \in \mathbf{ob} \mathcal{J}$.
- This entails that $\mathcal{V}\text{-CAT}(\mathcal{J}, \mathcal{C}) \simeq \mathbf{End}_{\mathcal{J}}(\mathcal{C})$, the ordinary category of \mathcal{J} -ary \mathcal{V} -endofunctors of \mathcal{C} , i.e. \mathcal{V} -endofunctors $H : \mathcal{C} \rightarrow \mathcal{C}$ that preserve left Kan extensions along j . We let $\mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ be the (ordinary) category of \mathcal{J} -ary \mathcal{V} -monads on \mathcal{C} .

Factories

- A *symmetric monoidal closed factory* is a symmetric monoidal closed category \mathcal{V} with an enriched factorization system $(\mathcal{E}, \mathcal{M})$ [8].
- If \mathcal{V} is a closed factory, then a \mathcal{V} -factory is a \mathcal{V} -category \mathcal{C} equipped with an enriched factorization system $(\mathcal{E}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}})$ that is compatible with $(\mathcal{E}, \mathcal{M})$, i.e. for every $C \in \mathbf{ob}\mathcal{C}$, the \mathcal{V} -functor $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathcal{V}$ preserves the right class.
- A \mathcal{V} -factory \mathcal{C} is *cocomplete* if \mathcal{C} is cocomplete and has arbitrary cointersections of \mathcal{E} -morphisms.

Boundedness

- Let α be a regular cardinal and let \mathcal{C} and \mathcal{D} be cocomplete \mathcal{V} -factegories. A \mathcal{V} -functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is α -bounded if for every α -filtered diagram $D : \mathcal{A} \rightarrow \mathcal{C}_0$ and every \mathcal{M} -cocone $m = (m_A : D_A \rightarrow C)_A$ on D , if $\overline{m} : \mathbf{colim} D \rightarrow C$ lies in \mathcal{E} , then $\overline{Fm} : \mathbf{colim} FD \rightarrow FC$ lies in \mathcal{E} (see [3]).
- If $C \in \mathbf{ob}\mathcal{C}$, then C is α -bounded if the \mathcal{V} -functor $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathcal{V}$ is α -bounded.
- If $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is a subcategory of arities in a \mathcal{V} -factegory \mathcal{C} , then \mathcal{J} is (α) -bounded if every $J \in \mathbf{ob}\mathcal{J}$ is (α) -bounded.

Blanket Assumptions

All of our results hold for any subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a \mathcal{V} -category \mathcal{C} satisfying the following assumptions:

- \mathcal{V} is a complete and cocomplete symmetric monoidal closed category.
- \mathcal{C} is a cocomplete and cotensored \mathcal{V} -category such that either $(\mathcal{E}, \mathcal{M})$ is proper or \mathcal{C} is \mathcal{E} -cowellpowered.
- $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is bounded and eleutheric.

Examples

- The subcategory of arities $j : \mathcal{C}_\alpha \hookrightarrow \mathcal{C}$ of α -presentable objects in a locally α -presentable \mathcal{V} -category \mathcal{C} over a locally α -presentable closed category \mathcal{V} . The \mathcal{J} -ary endofunctors are those that preserve α -filtered colimits (i.e. have rank α).
- The subcategory of arities $j : \mathcal{C}_\Phi \hookrightarrow \mathcal{C}$ consisting of suitable Φ -presentable objects in a locally Φ -presentable \mathcal{V} -category for a class of weights Φ satisfying Axiom A from [6] in a locally bounded and \mathcal{C} -cowellpowered \mathcal{V} . The \mathcal{J} -ary endofunctors are those that preserve Φ -flat colimits.

Examples

- The subcategory of arities $j : \{I\} \hookrightarrow \mathcal{V}$ consisting of just the unit object in a complete and cocomplete symmetric monoidal closed factegory \mathcal{V} . The \mathcal{J} -ary endofunctors are those of the form $X \otimes (-) : \mathcal{V} \rightarrow \mathcal{V}$ ($X \in \mathbf{ob}\mathcal{V}$).
- The subcategory of arities $j : \mathbb{N}_{\mathcal{V}} \hookrightarrow \mathcal{V}$ consisting of the finite copowers of the unit object in any symmetric monoidal closed π -category \mathcal{V} [1]. The \mathcal{J} -ary endofunctors are those that preserve $\mathbb{N}_{\mathcal{V}}$ -flat colimits (incl. sifted colimits).
- Any eleutheric subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in a locally bounded \mathcal{V} -category \mathcal{C} over a locally bounded closed category \mathcal{V} [9].

Signatures and Free \mathcal{J} -ary Monads

Let $j : \mathcal{J} \hookrightarrow \mathcal{C}$ be a subcategory of arities. A \mathcal{J} -signature in \mathcal{C} is just a functor $\Sigma : \mathbf{ob} \mathcal{J} \rightarrow \mathcal{C}_0$. We then have a monadic forgetful functor $V : \mathbf{End}_{\mathcal{J}}(\mathcal{C}) \rightarrow \mathbf{Sig}_{\mathcal{J}}(\mathcal{C})$. We have also shown that the forgetful functor $W : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow \mathbf{End}_{\mathcal{J}}(\mathcal{C})$ is monadic:

Theorem

If $H \in \mathbf{End}_{\mathcal{J}}(\mathcal{C})$, then $G : H\text{-Alg} \rightarrow \mathcal{C}$ has a left adjoint $F : \mathcal{C} \rightarrow H\text{-Alg}$, and the \mathcal{V} -monad GF is an algebraically free \mathcal{V} -monad on H , i.e. $GF\text{-Alg} \cong H\text{-Alg}$, which is moreover \mathcal{J} -ary. So $W : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow \mathbf{End}_{\mathcal{J}}(\mathcal{C})$ has a left adjoint and is monadic.

Presentations of \mathcal{J} -ary Monads

- Let $j : \mathcal{J} \hookrightarrow \mathcal{C}$ be a subcategory of arities. A \mathcal{J} -presentation is a pair of \mathcal{J} -signature morphisms $\alpha, \beta : \Gamma \rightrightarrows \mathbb{T}_\Sigma$, where Γ, Σ are \mathcal{J} -signatures and \mathbb{T}_Σ is the free \mathcal{J} -ary monad on Σ .
- We want to show that every \mathcal{J} -presentation P generates a \mathcal{J} -ary monad \mathbb{T}_P , i.e. that there is a coequalizer $\mathbb{T}_\Gamma \rightrightarrows \mathbb{T}_\Sigma \rightarrow \mathbb{T}_P$ in $\mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ which is preserved by the semantics functor $\mathbf{Alg} : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow (\mathcal{V}\mathbf{CAT}/\mathcal{C})^{\mathbf{op}}$.
- If $\mathbb{M} : \mathcal{K} \rightarrow \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ is a small diagram, then we will write $G^{\mathbb{M}} : \mathbb{M}\text{-Alg} \rightarrow \mathcal{C}$ for the limit of $\mathbf{Alg} \circ \mathbb{M}^{\mathbf{op}} : \mathcal{K}^{\mathbf{op}} \rightarrow \mathcal{V}\mathbf{CAT}/\mathcal{C}$.

Algebraic Colimits of \mathcal{J} -ary Monads

A colimit in $\mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ is *algebraic* if it is preserved by $\mathbf{Alg} : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow (\mathcal{V}\mathbf{CAT}/\mathcal{C})^{\text{op}}$.

Theorem

If $\mathbb{M} : \mathcal{K} \rightarrow \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ is a small diagram, then $G^{\mathbb{M}} : \mathbb{M}\text{-Alg} \rightarrow \mathcal{C}$ has a left adjoint $F^{\mathbb{M}} : \mathcal{C} \rightarrow \mathbb{M}\text{-Alg}$, and the \mathcal{V} -monad $G^{\mathbb{M}}F^{\mathbb{M}}$ is an algebraic colimit of \mathbb{M} , which is moreover \mathcal{J} -ary. So $\mathbf{Mnd}_{\mathcal{J}}(\mathcal{C})$ has small algebraic colimits. In particular, every \mathcal{J} -presentation generates a \mathcal{J} -ary monad.

Monadicity of \mathcal{J} -ary Monads

Using Lack's result [5, Theorem 2], we have also shown:

Theorem

The forgetful functor $U : \mathbf{Mnd}_{\mathcal{J}}(\mathcal{C}) \rightarrow \mathbf{Sig}_{\mathcal{J}}(\mathcal{C})$ is monadic.

Thus, every \mathcal{J} -ary monad on \mathcal{C} has a \mathcal{J} -presentation.

Locally Bounded \mathcal{V} -Categories

Theorem

Let \mathcal{C} be a locally bounded \mathcal{V} -category [9] over a locally bounded closed category \mathcal{V} . Then every subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ is bounded.

Thus, all of our results hold for any eleutheric subcategory of arities in any locally bounded \mathcal{V} -category \mathcal{C} over a locally bounded closed category \mathcal{V} .

Conclusions

- We have developed a very general framework for studying signatures and presentations of enriched monads for a subcategory of arities that includes locally bounded enriched categories and arbitrary π -categories.
- Specifically, we have seen that free \mathcal{J} -ary monads and presentations for \mathcal{J} -ary monads can be obtained for any bounded and eleutheric subcategory of arities $j : \mathcal{J} \hookrightarrow \mathcal{C}$ in any cocomplete and cotensored \mathcal{V} -factegory \mathcal{C} .
- In particular, our results apply to any eleutheric subcategory of arities in any locally bounded \mathcal{V} -category over a locally bounded closed category \mathcal{V} .

Thank you!

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