Covariant isotropy of Grothendieck toposes

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Introduction

- Covariant isotropy is a (recent) categorical construction that can be seen as providing a generalized notion of *conjugation* or *inner automorphism* for an arbitrary category.
- In previous work [3, 6, 4], we used techniques from categorical logic to characterize the covariant isotropy of any locally finitely presentable category $\mathbb{C}$, and in particular of any *presheaf category*.
- In this talk, we will provide an overview of covariant isotropy and show that its characterization for any presheaf category (essentially) extends to any Grothendieck topos. This is based on my recent preprint [7] *Covariant isotropy of Grothendieck toposes*: `https://arxiv.org/abs/2104.13487`

Motivation for covariant isotropy

- George Bergman proved in [1] that the inner automorphisms of groups can be characterized purely *categorically* as the group automorphisms that extend naturally along any group homomorphism.

- To see this, observe first that if α is an inner automorphism of a group G (induced by $s \in G$), then for each group morphism $f : G \to H$, we can extend α along f to define an inner automorphism

  $$\alpha_f : H \xrightarrow{\sim} H$$

  by conjugation with $f(s) \in H$ (so that $\alpha_{\mathbf{id}_G} = \alpha$).

Motivation

- This family of automorphisms $(\alpha_f)_f$ is *coherent*, in the sense that it satisfies the following naturality property: if $f : G \to G'$ and $f' : G' \to G''$ are group homomorphisms, then the following diagram commutes:

$$\begin{array}{ccc} G' & \xrightarrow{\alpha_f} & G' \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f'} \\ G'' & \xrightarrow[\alpha_{f' \circ f}]{} & G'' \end{array}$$

Bergman's theorem

For a group G, let us call an *arbitrary* family of automorphisms

$$\left(\alpha_f : \mathbf{cod}(f) \xrightarrow{\sim} \mathbf{cod}(f)\right)_{\mathbf{dom}(f)=G}$$

with the above naturality property an *extended inner automorphism* of G. Such a family is a natural automorphism of $G/\mathbf{Group} \to \mathbf{Group}$.

Theorem (Bergman [1])

Let G be a group and $\alpha : G \xrightarrow{\sim} G$ an automorphism of G. Then α is an **inner** *automorphism of G iff there is an extended inner automorphism $(\alpha_f)_f$ of G with $\alpha = \alpha_{\mathbf{id}_G}$.*

This provides a completely *categorical* characterization of inner automorphisms of groups: they are exactly those group automorphisms that are 'coherently extendible' along morphisms out of their domain.

Covariant isotropy

- We have a functor $\mathcal{Z} : \textbf{Group} \to \textbf{Group}$ that sends any group G to its group of extended inner automorphisms $\mathcal{Z}(G)$. We refer to $\mathcal{Z}$ as the *covariant isotropy group (functor)* of the category **Group**. (Bergman's theorem actually entails that $\mathcal{Z} \cong 1 : \textbf{Group} \to \textbf{Group}$.)

- In fact, *any* category $\mathbb{C}$ has a *covariant isotropy group (functor)*

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C} \to \textbf{Group}$$

that sends each object $C \in \mathbb{C}$ to the group of extended inner automorphisms of C, i.e. families of automorphisms

$$\left(\alpha_f : \textbf{cod}(f) \xrightarrow{\sim} \textbf{cod}(f)\right)_{\textbf{dom}(f)=C}$$

in $\mathbb{C}$ with the same naturality property as before, i.e. natural automorphisms of the projection functor $C/\mathbb{C} \to \mathbb{C}$.

Covariant isotropy

- We can also turn Bergman's characterization of inner automorphisms in **Group** into a *definition* of inner automorphisms in an arbitrary category $\mathbb{C}$: if $C \in \mathbb{C}$, we say that an automorphism $\alpha : C \xrightarrow{\sim} C$ is *inner* if there is an extended inner automorphism $(\alpha_f)_f \in \mathcal{Z}_{\mathbb{C}}(C)$ with $\alpha_{\mathbf{id}_C} = \alpha$.

- Notice that **Group** is the category of (set-based) *models* of an *algebraic theory*, i.e. a set of equational axioms between terms.

- In [3, 6, 4] we generalized ideas from the proof of Bergman's Theorem to give a logical characterization of the (extended) inner automorphisms of $\mathbb{T}$**mod**, i.e. of the covariant isotropy group of $\mathbb{T}$**mod**, for any finitary *quasi-equational* theory $\mathbb{T}$.

Quasi-equational theories

- A finitary quasi-equational theory $\mathbb{T}$ over a multi-sorted finitary equational signature Σ is a set of *implications* (the *axioms* of $\mathbb{T}$) of the form $\varphi \Rightarrow \psi$, with φ, ψ finitary Horn formulas (see [5]).
- The operation symbols of a quasi-equational theory are only required to be *partially* defined. If t is a term, we write $t \downarrow$ as an abbreviation for $t = t$, meaning 't is defined'.
- If λ is a regular cardinal, a λ-*ary* quasi-equational theory $\mathbb{T}$ allows for λ-ary operations and λ-ary conjunctions.

Examples

- Any algebraic theory is a finitary quasi-equational theory, as are the theories of categories, groupoids, strict monoidal categories, and presheaves on any small category.

- If $(\mathbb{C}, \mathcal{J})$ is a small site, then the Grothendieck topos $\mathbf{Sh}(\mathbb{C}, \mathcal{J})$ is the category of models for a λ-ary quasi-equational theory $\mathbb{T}^{(\mathbb{C},\mathcal{J})}$. The sorts are the objects of $\mathbb{C}$, for any arrow $f : C \to D$ of $\mathbb{C}$ there is a unary operation symbol $\widehat{f} : D \to C$, and for any covering sieve $J \in \mathcal{J}(C)$ there is an (infinitary) operation symbol $\sigma_J : \prod_{f \in J} \mathbf{dom}(f) \to C$.

- One then has axioms expressing functoriality and the fact that any matching family has a unique amalgamation, so that the models of $\mathbb{T}^{(\mathbb{C},\mathcal{J})}$ are exactly the sheaves on $(\mathbb{C}, \mathcal{J})$.

The isotropy group of a quasi-equational theory

- Fix a λ-ary quasi-equational theory $\mathbb{T}$ over a λ-ary signature Σ, and let $\mathbb{T}\mathbf{mod}$ be its category of (set-based) models.

- We will now review the *logical/syntactic* characterization of the covariant isotropy group

$$\mathcal{Z}_{\mathbb{T}} : \mathbb{T}\mathbf{mod} \to \mathbf{Group}$$

  of $\mathbb{T}\mathbf{mod}$. This was achieved for finitary $\mathbb{T}$ in [3, 6, 4] and extended to general λ-ary $\mathbb{T}$ in [7].

- Using the quasi-equational syntax of $\mathbb{T}$, we can define a notion of *definable automorphism* for a model M of $\mathbb{T}$, and the definable automorphisms of any $M \in \mathbb{T}\mathbf{mod}$ form a group $\mathbf{DefInn}(M)$.

Definable automorphisms

- If $\mathbb{T}$ is single-sorted, then given $M \in \mathbb{T}\mathbf{mod}$, one can form the $\mathbb{T}$-model $M\langle \mathbf{x} \rangle$ obtained from M by freely adjoining an indeterminate element $\mathbf{x}$. Elements of $M\langle \mathbf{x} \rangle$ are congruence classes $[t]$ of terms t involving $\mathbf{x}$ and constants from M, where two terms s, t are congruent if they are provably equal in the *diagram theory* $\mathbb{T}(M, \mathbf{x})$ of M extended by the axiom $\top \vdash \mathbf{x} \downarrow$.

- An element $[t] \in M\langle \mathbf{x} \rangle$ is *(substitutionally) invertible* if there is some $[s] \in M\langle \mathbf{x} \rangle$ with

$$\mathbb{T}(M, \mathbf{x}) \vdash t[s/\mathbf{x}] = \mathbf{x} = s[t/\mathbf{x}].$$

- If f is an n-ary operation symbol of Σ, then $[t] \in M\langle \mathbf{x} \rangle$ *commutes generically with f* if $\mathbb{T}(M, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ proves the sequent

$$f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \downarrow \vdash t[f(\mathbf{x}_1, \ldots, \mathbf{x}_n)/\mathbf{x}] = f(t(\mathbf{x}_1), \ldots, t(\mathbf{x}_n)).$$

Definable automorphisms

- If f is an n-ary operation symbol of Σ, then $[t] \in M\langle \mathbf{x} \rangle$ *reflects definedness of f* if $\mathbb{T}(M, \mathbf{x}_1, \ldots, \mathbf{x}_n)$ proves the sequent

$$t[f(\mathbf{x}_1, \ldots, \mathbf{x}_n)/\mathbf{x}] \downarrow \vdash f(\mathbf{x}_1, \ldots, \mathbf{x}_n) \downarrow .$$

- We define $\mathbf{DefInn}(M)$ to be the group of all elements $[t] \in M\langle \mathbf{x} \rangle$ that are substitutionally invertible and commute generically with and reflect definedness of every operation symbol of Σ.

- If $\mathbb{T}$ is multi-sorted, one can extend the above definitions appropriately.

The isotropy group of a quasi-equational theory

Theorem ([4, 7])

Let $\mathbb{T}$ be a λ-ary quasi-equational theory. For any $M \in \mathbb{T}\mathbf{mod}$, the covariant isotropy group $\mathcal{Z}_{\mathbb{T}}(M)$, i.e. the group of extended inner automorphisms of M, is isomorphic to the group $\mathbf{DefInn}(M)$ of definable automorphisms of M.

In [3] we used this result to show that the categorical inner automorphisms in many categories of algebraic structures (monoids, (abelian) groups, non-commutative unital rings, etc.) are precisely the conjugation-theoretic inner automorphisms.

Presheaf categories

- In [4] we also characterized the covariant isotropy group of a presheaf category $\mathbf{Set}^{\mathbb{C}}$ for a small category $\mathbb{C}$.

- If $F : \mathbb{C} \to \mathbf{Set}$ is a presheaf, we showed that $\mathbf{DefInn}(F)$ consists (up to isomorphism) of exactly the natural automorphisms $\alpha : F \xrightarrow{\sim} F$ induced by some element $\psi \in \mathbf{Aut}\,(1_{\mathbb{C}})$, in the sense that

$$(C \in \mathbb{C}) \qquad \alpha_C = F(\psi_C) : F(C) \xrightarrow{\sim} F(C).$$

- It then follows that the covariant isotropy group $\mathcal{Z} : \mathbf{Set}^{\mathbb{C}} \to \mathbf{Group}$ is *constant* on the automorphism group $\mathbf{Aut}(1_{\mathbb{C}})$ of $1_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$.

Grothendieck toposes

- In [7], we wanted to determine if this result extended to arbitrary sheaf categories.
- For convenient technical reasons, we first restricted our attention to subcanonical sites $(\mathbb{C}, \mathcal{J})$ where no object is covered by the empty sieve. If F is a sheaf over any such site, we showed that $\mathbf{DefInn}(F)$ consists (up to isomorphism) of precisely those natural automorphisms $\alpha : F \xrightarrow{\sim} F$ induced by some element $\psi \in \mathbf{Aut}\,(1_{\mathbb{C}})$, as above. The proof of this fact is the most non-trivial part of the overall result in [7].

Grothendieck toposes

- Hence, as for presheaf categories, if $(\mathbb{C}, \mathcal{J})$ is any small subcanonical site in which no object is covered by the empty sieve, it follows that the covariant isotropy group $\mathcal{Z} : \mathbf{Sh}(\mathbb{C}, \mathcal{J}) \to \mathbf{Group}$ is constant on the automorphism group $\mathbf{Aut}\,(1_{\mathbb{C}})$ of $1_{\mathbb{C}}$.

- We now want to remove the assumptions of subcanonicity and no object being covered by the empty sieve. The second property is easier to remove: if $(\mathbb{C}, \mathcal{J})$ is any small subcanonical site, one can find another subcanonical site $(\mathbb{D}, \mathcal{K})$ in which no object is covered by the empty sieve, with $\mathbf{Sh}(\mathbb{C}, \mathcal{J}) \simeq \mathbf{Sh}(\mathbb{D}, \mathcal{K})$ and $\mathbf{Aut}\,(1_{\mathbb{C}}) \cong \mathbf{Aut}\,(1_{\mathbb{D}})$.

- So if $(\mathbb{C}, \mathcal{J})$ is any small subcanonical site, then $\mathcal{Z} : \mathbf{Sh}(\mathbb{C}, \mathcal{J}) \to \mathbf{Group}$ is still constant on $\mathbf{Aut}\,(1_{\mathbb{C}})$.

Grothendieck toposes

- We now want to consider arbitrary small sites $(\mathbb{C}, \mathcal{J})$. First, if $\mathcal{E}$ is a (locally small) category with small full dense subcategory $\mathbb{C} \hookrightarrow \mathcal{E}$, then $\mathbf{Aut}\,(1_{\mathcal{E}}) \cong \mathbf{Aut}\,(1_{\mathbb{C}})$.
- Now if $(\mathbb{C}, \mathcal{J})$ is any small site, then there is a subcanonical topology $\mathcal{K}$ on the small full dense subcategory $\mathbf{ay}\mathbb{C} \hookrightarrow \mathbf{Sh}(\mathbb{C}, \mathcal{J})$ for which $\mathbf{Sh}(\mathbb{C}, \mathcal{J}) \simeq \mathbf{Sh}(\mathbf{ay}\mathbb{C}, \mathcal{K})$.
- So then $\mathcal{Z} : \mathbf{Sh}(\mathbb{C}, \mathcal{J}) \to \mathbf{Group}$ is constant on $\mathbf{Aut}\,(1_{\mathbf{ay}\mathbb{C}}) \cong \mathbf{Aut}\,\left(1_{\mathbf{Sh}(\mathbb{C}, \mathcal{J})}\right)$. In particular, if $(\mathbb{C}, \mathcal{J})$ is subcanonical, then $\mathbf{Aut}\,\left(1_{\mathbf{Sh}(\mathbb{C}, \mathcal{J})}\right) \cong \mathbf{Aut}\,(1_{\mathbb{C}})$, providing a characterization of the centre of $\mathbf{Sh}(\mathbb{C}, \mathcal{J})$.

Grothendieck toposes

- In particular, since $\mathbf{Set}^{\mathbb{C}^{\mathbf{op}}} = \mathbf{Sh}(\mathbb{C}, T)$ for the trivial (subcanonical) topology T on $\mathbb{C}$ (only maximal sieves cover), we recover our earlier result for presheaf toposes.

- If $(\mathbb{C}, \mathcal{J})$ is *not* subcanonical, there is in general no relation between $\mathbf{Aut}\,(1_{\mathbb{C}})$ and $\mathbf{Aut}\,(1_{\mathbf{ay}\mathbb{C}})$. E.g. if $\mathbf{Aut}\,(1_{\mathbb{C}})$ is non-trivial and $\mathcal{J}$ is such that *every* sieve covers, then $\mathcal{J}$ is not subcanonical and $\mathbf{Sh}(\mathbb{C}, \mathcal{J})$ is trivial, so that $\mathbf{Aut}\,(1_{\mathbf{ay}\mathbb{C}}) \cong \mathbf{Aut}\,\left(1_{\mathbf{Sh}(\mathbb{C},\mathcal{J})}\right)$ is trivial.

- Our result illustrates a major difference between covariant isotropy $\mathbf{Sh}(\mathbb{C}, \mathcal{J}) \to \mathbf{Group}$ and *contravariant* isotropy $\mathbf{Sh}(\mathbb{C}, \mathcal{J})^{\mathbf{op}} \to \mathbf{Group}$ (cf. [2]) for Grothendieck toposes: while the latter is always *representable* by a sheaf of groups $Z : \mathbb{C}^{\mathbf{op}} \to \mathbf{Group}$, the former is always *constant* (on the global sections of Z).

Conclusions

- Via Bergman's purely *categorical* characterization of the inner automorphisms of groups, covariant isotropy can be regarded as providing a notion of *conjugation* or *inner automorphism* for arbitrary categories.

- We have characterized the covariant isotropy group of $\mathbb{T}\mathbf{mod}$ for any λ-ary quasi-equational theory $\mathbb{T}$: we have $\mathcal{Z}_{\mathbb{T}}(M) \cong \mathbf{DefInn}(M)$ for any $M \in \mathbb{T}\mathbf{mod}$.

- Using this result, we have shown that the characterization of covariant isotropy for presheaf toposes (essentially) extends to all Grothendieck toposes: for a small subcanonical site $(\mathbb{C}, \mathcal{J})$, the covariant isotropy group $\mathcal{Z} : \mathbf{Sh}(\mathbb{C}, \mathcal{J}) \to \mathbf{Group}$ is constant on $\mathbf{Aut}\,(1_{\mathbb{C}})$.

Thank you!

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