

Locally bounded enriched categories

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Introduction

- Locally bounded ordinary categories were (implicitly) introduced by Freyd and Kelly in
 - ▶ [3] P.J. Freyd and G.M. Kelly. Categories of continuous functors I. *Journal of Pure and Applied Algebra* Vol. 2, Issue 3, 169-191, 1972.

as a context for proving reflectivity results for orthogonal subcategories and categories of models.

- The notion of locally bounded (symmetric monoidal closed) category was then explicitly defined by Kelly in [4, Chapter 6] and used as the basis for a general treatment of enriched limit theories.

Introduction

- Locally bounded categories subsume locally presentable categories and many “topological” categories that are *not* locally presentable.
- Speaking of locally presentable categories, in
 - ▶ [5] G.M. Kelly. Structures defined by finite limits in the enriched context I. *Cahiers de Topologie et Géométrie Catégoriques Différentielle* 23, No. 1, 3-42, 1982.

Kelly defined the notion of a locally presentable \mathcal{V} -category over a locally presentable closed category \mathcal{V} .

- Kelly *did* define the notion of a locally bounded closed category \mathcal{V} in [4, Chapter 6], but never got around to defining the notion of a locally bounded \mathcal{V} -category over such a \mathcal{V} . That’s where this talk comes in!

Locally bounded (ordinary) categories

- Let's start by reviewing the definition of a locally bounded (ordinary) category. A **(proper) factory** is a category \mathcal{C} with a proper factorization system $(\mathcal{E}, \mathcal{M})$. The factory \mathcal{C} is **cocomplete** if \mathcal{C} is cocomplete and has arbitrary cointersections (i.e. wide pushouts) of \mathcal{E} -morphisms.
- Given a small \mathcal{M} -sink $(m_i : C_i \rightarrow C)_{i \in I}$ in \mathcal{C} , its **union** is the \mathcal{M} -subobject m obtained from the $(\mathcal{E}, \mathcal{M})$ -factorization

$$\prod_i C_i \xrightarrow{e} \bigcup_i C_i \xrightarrow{m} C.$$

The sink $(m_i)_i$ is α -**filtered** if any sub-sink of size $< \alpha$ factors through some m_i .

Locally bounded (ordinary) categories

- A functor $U : \mathcal{C} \rightarrow \mathcal{D}$ between cocomplete factegories that preserves \mathcal{M} is said to **preserve** (α -**filtered**) \mathcal{M} -**unions** if for any (α -filtered) \mathcal{M} -sink $(m_i)_i$ with union m , Um is the union of the \mathcal{M} -sink $(Um_i)_i$. If U preserves \mathcal{M} and preserves α -filtered \mathcal{M} -unions, we also say that U is α -**bounded**.
- In particular, an object $C \in \mathbf{ob}\mathcal{C}$ of a cocomplete factegory \mathcal{C} is α -bounded if $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathbf{Set}$ preserves α -filtered \mathcal{M} -unions.
- Finally, a set $\mathcal{G} \subseteq \mathbf{ob}\mathcal{C}$ of a cocomplete factegory is an $(\mathcal{E}, \mathcal{M})$ -**generator** if for any $C \in \mathbf{ob}\mathcal{C}$, the canonical morphism

$$\coprod_{G \in \mathcal{G}} \mathcal{C}(G, C) \cdot G \rightarrow C$$

lies in \mathcal{E} (equivalently, the functors $\mathcal{C}(G, -) : \mathcal{C} \rightarrow \mathbf{Set}$ ($G \in \mathcal{G}$) are **jointly \mathcal{M} -conservative**).

Locally bounded (ordinary) categories

Definition (Kelly [4])

A locally α -bounded category is a cocomplete category \mathcal{C} with an $(\mathcal{E}, \mathcal{M})$ -generator consisting of α -bounded objects.

Note the parallel with locally α -presentable categories: a locally α -presentable category is a cocomplete category \mathcal{C} with a *strong* generator consisting of *α -presentable* objects.

Examples

- Any locally α -presentable category [3, 3.2.3], with $(\mathcal{E}, \mathcal{M}) = (\mathbf{StrongEpi}, \mathbf{Mono})$ and the given strong generator of α -presentable (and hence α -bounded) objects.
- Any topological category over \mathbf{Set} is locally \aleph_0 -bounded [8, 2.3], with $(\mathcal{E}, \mathcal{M}) = (\mathbf{Epi}, \mathbf{StrongMono})$ and the generator consisting of just the discrete object on $\{*\}$.
- Any cocomplete locally cartesian closed category (e.g. elementary quasitopos) with a generator and arbitrary cointersections of epimorphisms, so that $(\mathcal{E}, \mathcal{M}) = (\mathbf{Epi}, \mathbf{StrongMono})$. These include the concrete quasitoposes of Dubuc [2].

Locally bounded closed categories

We now recall Kelly's definition of locally bounded symmetric monoidal closed category:

Definition (Kelly [4])

A symmetric monoidal closed category \mathcal{V} is **locally α -bounded as a closed category** if \mathcal{V}_0 is locally α -bounded, the proper factorization system $(\mathcal{E}, \mathcal{M})$ is enriched, the unit object $I \in \mathbf{ob}\mathcal{V}$ is α -bounded, and $G \otimes G'$ is α -bounded for all $G, G' \in \mathcal{G}$.

For example: any symmetric monoidal closed category \mathcal{V} with \mathcal{V}_0 locally α -presentable [4, Chapter 6]; any topological category over **Set**; any cocomplete locally cartesian closed category with generator and arbitrary epi-cointersections (e.g. any concrete quasitopos).

\mathcal{V} -factegories

- For the remainder of the talk, \mathcal{V} will be a locally α -bounded closed category (sometimes a stronger assumption than needed).
- An enriched factorization system $(\mathcal{E}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}})$ on a \mathcal{V} -category \mathcal{C} [7] is **compatible** with $(\mathcal{E}, \mathcal{M})$ if each $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathcal{V}$ ($C \in \mathbf{ob}\mathcal{C}$) preserves the right class.
- A \mathcal{V} -**factegory** is a \mathcal{V} -category \mathcal{C} with an enriched proper factorization system $(\mathcal{E}_{\mathcal{C}}, \mathcal{M}_{\mathcal{C}})$ that is compatible with $(\mathcal{E}, \mathcal{M})$. The \mathcal{V} -factegory \mathcal{C} is **cocomplete** if the \mathcal{V} -category \mathcal{C} is cocomplete and has arbitrary (conical) cointersections of \mathcal{E} -morphisms.

Enriched $(\mathcal{E}, \mathcal{M})$ -generators

- Let \mathcal{C} be a cocomplete \mathcal{V} -category. A set $\mathcal{G} \subseteq \mathbf{ob}\mathcal{C}$ is an **enriched $(\mathcal{E}, \mathcal{M})$ -generator** if for each $C \in \mathbf{ob}\mathcal{C}$, the canonical morphism $\coprod_{G \in \mathcal{G}} \mathcal{C}(G, C) \otimes G \rightarrow C$ lies in \mathcal{E} .
- A set $\mathcal{G} \subseteq \mathbf{ob}\mathcal{C}$ is an enriched $(\mathcal{E}, \mathcal{M})$ -generator iff the representable \mathcal{V} -functors $\mathcal{C}(G, -) : \mathcal{C} \rightarrow \mathcal{V}$ ($G \in \mathcal{G}$) are **jointly \mathcal{M} -conservative**.

Enriched α -bounded objects

Let \mathcal{C} be a cocomplete \mathcal{V} -category. An object $C \in \mathbf{ob}\mathcal{C}$ is an **enriched α -bounded object** if $\mathcal{C}(C, -) : \mathcal{C} \rightarrow \mathcal{V}$ preserves α -filtered \mathcal{M} -unions.

Definition

A **locally α -bounded \mathcal{V} -category** is a cocomplete \mathcal{V} -category \mathcal{C} with an enriched $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{G} consisting of enriched α -bounded objects.

- Note the parallel with locally α -presentable \mathcal{V} -categories: a locally α -presentable \mathcal{V} -category is a cocomplete \mathcal{V} -category with an enriched *strong* generator of enriched α -presentable objects [5, 3.1].
- If \mathcal{V} is a locally α -bounded closed category with ordinary $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{G} , then \mathcal{V} is itself a locally α -bounded \mathcal{V} -category with enriched $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{G} .
- Any locally bounded \mathcal{V} -category is total and complete.

Bounding right adjoints

The following notion of *bounding right adjoint* is fundamental for constructing examples of locally bounded \mathcal{V} -categories:

Definition

Let $U : \mathcal{C} \rightarrow \mathcal{D}$ be a \mathcal{V} -functor between cocomplete \mathcal{V} -categories. Then U is an α -**bounding right adjoint** if U is α -bounded and has a left adjoint whose counit is pointwise in \mathcal{E} .

U is an α -bounding right adjoint iff U is α -bounded, has a left adjoint, and is \mathcal{M} -conservative, iff U is α -bounded, has a left adjoint, and reflects \mathcal{E} . An α -bounding right adjoint is automatically $(\mathcal{V}$ -)faithful.

Bounding right adjoints

Theorem

Let \mathcal{C} be a cocomplete \mathcal{V} -factegory and let $\mathcal{G} \subseteq \mathbf{ob}\mathcal{C}$ be a set. Then \mathcal{C} is locally α -bounded with enriched $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{G} iff the nerve $\mathbf{y}_{\mathcal{G}} : \mathcal{C} \rightarrow [\mathcal{G}^{\mathbf{op}}, \mathcal{V}]$ is an α -bounding right adjoint.

Theorem

Let \mathcal{D} be a locally α -bounded \mathcal{V} -category and \mathcal{C} a cocomplete \mathcal{V} -factegory. If $U : \mathcal{C} \rightarrow \mathcal{D}$ is an α -bounding right adjoint, then \mathcal{C} is locally α -bounded.

Bounding right adjoints

Theorem

Let \mathcal{C} be a cocomplete \mathcal{V} -category. Then \mathcal{C} is locally α -bounded iff there exists a small \mathcal{V} -category \mathcal{A} and an α -bounding right adjoint $U : \mathcal{C} \rightarrow [\mathcal{A}, \mathcal{V}]$, i.e. a \mathcal{V} -functor $U : \mathcal{C} \rightarrow [\mathcal{A}, \mathcal{V}]$ that is α -bounded, \mathcal{M} -conservative, and has a left adjoint.

Note the parallel with Kelly's result [4, 3.1]: a cocomplete \mathcal{V} -category \mathcal{C} is locally α -presentable iff there exists a small \mathcal{V} -category \mathcal{A} and a \mathcal{V} -functor $U : \mathcal{C} \rightarrow [\mathcal{A}, \mathcal{V}]$ that has rank α , is conservative, and has a left adjoint.

Corollary: if \mathcal{C} is locally α -bounded and \mathcal{A} is small, then $[\mathcal{A}, \mathcal{C}]$ is locally α -bounded.

Enriched vs. ordinary local boundedness

Recall that \mathcal{V} is a locally α -bounded closed category with ordinary $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{G} .

Theorem

If \mathcal{C} is locally α -bounded with enriched $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{H} , then \mathcal{C}_0 is locally α -bounded with ordinary $(\mathcal{E}, \mathcal{M})$ -generator $\mathcal{G} \otimes \mathcal{H}$.

Theorem

If \mathcal{C} is a cocomplete \mathcal{V} -category such that \mathcal{C}_0 is locally bounded with ordinary $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{H} , then \mathcal{C} is locally bounded with enriched $(\mathcal{E}, \mathcal{M})$ -generator \mathcal{H} .

A representability theorem

It is well known that if \mathcal{C} is a locally presentable (even accessible) category, then a functor $U : \mathcal{C} \rightarrow \mathbf{Set}$ is *representable* iff U is continuous and has rank. We have a similar result for locally bounded categories:

Theorem

Let \mathcal{C} be a locally bounded and \mathcal{E} -cowellpowered \mathcal{V} -category. If $U : \mathcal{C} \rightarrow \mathcal{V}$ preserves \mathcal{M} , then U is representable iff U is continuous and bounded.

Adjoint functor theorems

Recall that a functor $U : \mathcal{C} \rightarrow \mathcal{D}$ between locally presentable categories has a left adjoint iff U is continuous and has rank.

Theorem

Let \mathcal{C}, \mathcal{D} be locally bounded \mathcal{V} -categories such that \mathcal{C} is \mathcal{E} -cowellpowered. If $U : \mathcal{C} \rightarrow \mathcal{D}$ preserves \mathcal{M} , then U has a left adjoint iff U is continuous and bounded.

Recall that if \mathcal{C} is locally presentable and \mathcal{D} arbitrary, then $F : \mathcal{C} \rightarrow \mathcal{D}$ has a right adjoint iff F is cocontinuous.

Theorem

Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a \mathcal{V} -functor from a locally bounded \mathcal{V} -category \mathcal{C} to an arbitrary \mathcal{V} -category \mathcal{D} . Then F has a right adjoint iff F is cocontinuous.

(In fact, \mathcal{C} just needs to be cocomplete \mathcal{V} -category with enriched $(\mathcal{E}, \mathcal{M})$ -generator.)

α -bounded-small limits

- It is well known that α -small limits commute with α -filtered colimits in any locally α -presentable category.
- If \mathcal{V} is a locally α -presentable closed category, then Kelly defined in [5, 4.1] the notion of an α -small weight $W : \mathcal{B} \rightarrow \mathcal{V} : |\mathbf{ob}\mathcal{B}| < \alpha$, $\mathcal{B}(B, B') \in \mathcal{V}_\alpha$ for all $B, B' \in \mathbf{ob}\mathcal{B}$, and $WB \in \mathcal{V}_\alpha$ for all $B \in \mathbf{ob}\mathcal{B}$.
- He then showed in [5, 4.9] that α -small weighted limits commute with conical α -filtered colimits in any locally α -presentable \mathcal{V} -category.
- If \mathcal{V} is a locally α -bounded closed category, we can define the similar notion of an α -bounded-small weight $W : \mathcal{B} \rightarrow \mathcal{V}$.

α -bounded-small limits

Definition

Let \mathcal{V} be a locally α -bounded closed category. A weight $W : \mathcal{B} \rightarrow \mathcal{V}$ is **α -bounded-small** if $|\mathbf{ob}\mathcal{B}| < \alpha$, $\mathcal{B}(B, B')$ is an enriched α -bounded object of \mathcal{V} for all $B, B' \in \mathbf{ob}\mathcal{B}$, and WB is an enriched α -bounded object of \mathcal{V} for all $B \in \mathbf{ob}\mathcal{B}$.

Kelly showed in [5, 4.3] that the saturation of the class of α -small weights is equal to the saturation of the class of weights for α -small conical limits and α -presentable cotensors. We similarly have:

Theorem

The saturation of the class of α -bounded-small weights is equal to the saturation of the class of weights for α -small conical limits and α -bounded cotensors.

α -bounded-small limits

Definition

Let \mathcal{C} be a complete and cocomplete \mathcal{V} -category and $W : \mathcal{B} \rightarrow \mathcal{V}$ a small weight. Then **W -limits commute with α -filtered \mathcal{M} -unions in \mathcal{C}** if the W -limit \mathcal{V} -functor $\{W, -\} : [\mathcal{B}, \mathcal{C}] \rightarrow \mathcal{C}$ is α -bounded.

Theorem

If \mathcal{C} is a locally α -bounded \mathcal{V} -category, then α -bounded-small weighted limits commute with α -filtered \mathcal{M} -unions in \mathcal{C} .

Reflectivity and local boundedness

- Freyd and Kelly proved in [3, 4.1.3, 4.2.2] that if \mathcal{C} is an \mathcal{E} -cowellpowered locally bounded ordinary category and Θ is a “quasi-small” class of morphisms in \mathcal{C} , then the orthogonal subcategory $\Theta^\perp \hookrightarrow \mathcal{C}$ is reflective and locally bounded.
- Kelly showed in [4, Chapter 6] that the reflectivity still holds even without \mathcal{E} -cowellpoweredness.

We have enriched both results as follows:

Theorem

Let \mathcal{C} be a locally bounded \mathcal{V} -category with a “quasi-small” class of morphisms Θ . Then the enriched orthogonal sub- \mathcal{V} -category $\Theta^{\perp\mathcal{V}} \hookrightarrow \mathcal{C}$ is reflective, and $\Theta^{\perp\mathcal{V}}$ is locally bounded if \mathcal{C} is \mathcal{E} -cowellpowered.

Reflectivity and local boundedness

Freyd and Kelly also proved in [3, 5.2.1, 5.2.2] that if \mathcal{C} is a locally bounded and \mathcal{E} -cowellpowered ordinary category and (\mathcal{A}, Φ) is a limit sketch, then $\Phi\text{-Mod}(\mathcal{A}, \mathcal{C})$ is reflective in $[\mathcal{A}, \mathcal{C}]$ and locally bounded.

Theorem

Let \mathcal{C} be a locally α -bounded \mathcal{V} -category and (\mathcal{A}, Φ) an enriched limit sketch [4, 6.3]. Then the full sub- \mathcal{V} -category $\Phi\text{-Mod}(\mathcal{A}, \mathcal{C}) \hookrightarrow [\mathcal{A}, \mathcal{C}]$ is reflective, and $\Phi\text{-Mod}(\mathcal{A}, \mathcal{C})$ is also locally bounded if \mathcal{C} is \mathcal{E} -cowellpowered. If every weight in Φ is α -bounded-small, then $\Phi\text{-Mod}(\mathcal{A}, \mathcal{C})$ is in fact locally α -bounded.

In particular, if \mathcal{T} is a Φ -theory for a class of small weights Φ , then $\Phi\text{-Cts}(\mathcal{T}, \mathcal{C})$ is reflective in $[\mathcal{T}, \mathcal{C}]$, and is locally bounded if \mathcal{C} is \mathcal{E} -cowellpowered.

Reflectivity and local boundedness

As a corollary, we obtain the following result for the enriched algebraic theories of Lucyshyn-Wright [6]:

Theorem

Let $\mathcal{J} \hookrightarrow \mathcal{V}$ be a small system of arities, let \mathcal{T} be a \mathcal{J} -theory, and let \mathcal{C} be a locally bounded and \mathcal{E} -cowellpowered \mathcal{V} -category. Then the full sub- \mathcal{V} -category $\mathcal{T}\text{-Alg}(\mathcal{C}) \hookrightarrow [\mathcal{T}, \mathcal{C}]$ of the \mathcal{T} -algebras is reflective and locally bounded, and the forgetful \mathcal{V} -functor $U^{\mathcal{T}} : \mathcal{T}\text{-Alg}(\mathcal{C}) \rightarrow \mathcal{C}$ is monadic.

In particular, if \mathcal{C} is a locally α -bounded and \mathcal{E} -cowellpowered ordinary category and \mathcal{T} is a Lawvere theory, then the category $\mathcal{T}\text{-Alg}(\mathcal{C})$ of \mathcal{T} -algebras in \mathcal{C} is reflective in $[\mathcal{T}, \mathcal{C}]$ and locally α -bounded, and the forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-Alg}(\mathcal{C}) \rightarrow \mathcal{C}$ is monadic.

In summary...

- We have defined a notion of locally bounded \mathcal{V} -category over a locally bounded closed category \mathcal{V} , which enriches the locally bounded ordinary categories of Freyd and Kelly, and parallels Kelly's notion of locally presentable \mathcal{V} -category over a locally presentable closed category \mathcal{V} .
- Examples of locally bounded closed categories include locally presentable closed categories, topological categories over **Set**, and epi-cocomplete quasitoposes with generators.
- Many of the results for locally presentable enriched categories have analogues for locally bounded enriched categories: representability theorems, adjoint functor theorems, and commutation of suitably small limits with suitably filtered colimits/unions.

In summary...

- Moreover, locally bounded enriched categories admit full enrichments of Freyd and Kelly's reflectivity results for orthogonal subcategories and categories of models.
- Lucyshyn-Wright and I have also shown that locally bounded enriched categories provide a fruitful setting for obtaining results on free monads, presentations of monads, and algebraic colimits of monads for a subcategory of arities (to be presented in forthcoming work).

α -bounded monads?

- One topic I did *not* touch on is the local boundedness of Eilenberg-Moore categories. It is (well) known that if \mathcal{C} is a locally α -presentable \mathcal{V} -category and \mathbb{T} is a \mathcal{V} -monad on \mathcal{C} with rank α , then the Eilenberg-Moore \mathcal{V} -category $\mathbb{T}\text{-Alg}$ is locally α -presentable, and $U^{\mathbb{T}} : \mathbb{T}\text{-Alg} \rightarrow \mathcal{C}$ is continuous and has rank α (see [1, 6.9]).
- Does an analogous result hold for α -bounded \mathcal{V} -monads on locally α -bounded \mathcal{V} -categories? Essentially yes, but with some slight subtleties/complications (to be presented in forthcoming work).

Thank you!

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