DUALITY BETWEEN CUBES AND BIPOINTED SETS

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Abstract

In light of the new model of intensional Martin-Löf type theory in the category of cubical sets ([4]), in this thesis we prove that an expanded version of the cube category (which we call the *Cartesian* cube category) is dual to the apparently simpler category of finite strictly bipointed sets. This entails that cubical sets (presheaves on the cube category) are equivalent to the apparently much simpler covariant presheaves on such finite bipointed sets. This duality of basic categories also allows us to show that the Cartesian category of cubes is the free finite-product category on an interval. Finally, we show that the category of finite *weakly* bipointed sets is dual to the free finite-*limit* category on an interval.

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Chapter 1

Introduction

Homotopy type theory (HoTT) is a new proof-relevant foundational system of logic that derives its name from the fact that it can be soundly modeled in homotopy theory ([11], [3]). The basic concept of (homotopy) type theory is that of a term t being of type A, which we write as

$$t : A,$$

and which may be intuitively read as 't is a proof of the proposition A' or as 't is a point of the space A'. Given two terms x, y of type A, one may then form the identity type $\mathrm{Id}_A(x, y)$ of x and y in A. A term

$$p : \mathrm{Id}_A(x, y)$$

can be intuitively seen as a proof that x and y are equal, or as a *path* or *identification* between x and y. But we can have more than just identifications between terms of a type A: we can also have identifications between identifications, and identifications between identifications between identifications, and so on *ad infinitum*. In other words, for any type A, we can have terms of the following kind:

$$x, y : A,$$

$$p, q : \mathrm{Id}_A(x, y),$$

$$r, s : \mathrm{Id}_{\mathrm{Id}_A(x,y)}(p, q), \ldots$$

In this way, every type A in HoTT inherits an infinite-dimensional structure given by the identity types over A. Under a simplified view of the traditional homotopy-theoretic model of HoTT in simplicial sets ([12]), types are modeled as (topological) spaces, terms as points of such spaces, and identifications between terms as paths between such points in such spaces.

In recent years a new constructive model of HoTT has been given in the category of cubical sets, which are presheaves on the cube category, whose objects can be regarded as abstract higher-dimensional or 'hyper' cubes. One of the main intuitive motivations for modeling HoTT in terms of cubical sets may be described as follows.

First, we may regard the elements or inhabitants of a type as simply points or 0-cubes of the type. Then, for any 0-cubes $x, y : A$, we may view a term $p : \mathrm{Id}_A(x, y)$ as a 1-cube or one-dimensional line between x and y in the type A. Next, for any 1-cubes $p : \mathrm{Id}_A(x, y)$ and $q : \mathrm{Id}_A(z, w)$ (for $z, w : A$), we may first form the type

$$X \doteq \Sigma_{a,b:A}\mathrm{Id}_A(a, b),$$

and then the type $Id_X(p, q)$. Then we can regard a term $r : Id_X(p, q)$ as a 2-cube or two-dimensional square linking the 1-cubes p and q, as in the following picture:

$$\begin{array}{ccc} x & \overset{r}{\rightsquigarrow} & z \\ {\scriptstyle p}\wr & & \wr{\scriptstyle q} \\ y & \overset{r}{\rightsquigarrow} & w \end{array}$$

Clearly, this intuitive interpretation can be carried on *ad infinitum* to all higher dimensions. In this way, the cubical point of view lends itself very nicely to the interpretation of the higher-dimensional structure of HoTT given by the identity type, as explored in [4, 5].

However, as the site of the cubical sets, the cube category can be difficult to work with in practice because the objects are abstract and the maps are determined formally as composites of basic maps, as will be made clear in the next chapter. The central result of this thesis will be that by adding a few more kinds of basic maps to the classical cube category, we get a new and expanded cube category $\mathbb{C}$, the *Cartesian* cube category, that is dual to the apparently *simpler* category $\mathcal{B}$ of finite bipointed sets, whose objects are just finite sets equipped with two distinct basepoints, and whose maps are just the basepoint-preserving functions. More succinctly, the central result of this thesis (Theorem 3.7) is that

$$\mathbb{C}^{op} \cong \mathcal{B}.$$

As a corollary with importance for cubical homotopy theory and modeling HoTT in cubical sets, we find that cubical sets (presheaves on $\mathbb{C}$) are equivalent to the apparently simpler covariant functors on finite bipointed sets. In other words, we obtain the following equivalence:

$$\mathrm{Sets}^{\mathbb{C}^{op}} \simeq \mathrm{Sets}^{\mathcal{B}}.$$

The proof that $\mathbb{C}^{op} \cong \mathcal{B}$ makes special use of a new Stone-type duality from a dualizing object. As detailed in Chapter 3, this duality between $\mathbb{C}$ and $\mathcal{B}$ is given by a two-element set as dualizing object, equipped with different structures in the two different categories $\mathbb{C}$ and $\mathcal{B}$. This central duality result also has an interesting consequence due to Lawvere duality ([9]). As we will discuss in more detail in Chapter 3, Lawvere duality roughly states that the syntax of an algebraic theory is dual to its semantics:

$$\text{Syntax} \simeq \text{Semantics}^{op}.$$

More precisely, given any algebraic theory $\mathcal{T}$, we can form the syntactic category $\mathcal{C}_{\mathcal{T}}$ of $\mathcal{T}$, whose objects are finite lists of variables and whose morphisms are tuples of terms-in-context related by provable equality in the theory $\mathcal{T}$. Moreover, we can form the category of finitely-generated free models (in Sets) of the theory $\mathcal{T}$, and then dualize it, to obtain the category $\text{Mod}_{\text{fgf}}(\mathcal{T})^{op}$. One of the central results of Lawvere duality theory then states that these two categories are equivalent, i.e. that

$$\mathcal{C}_{\mathcal{T}} \simeq \text{Mod}_{\text{fgf}}(\mathcal{T})^{op}.$$

Now, if we take $\mathcal{T}$ to be the simple algebraic theory with two constants and no equations, then (as we will easily show), the category $\mathcal{B}$ of finite strictly bipointed sets *is just* the category $\text{Mod}_{\text{fgf}}(\mathcal{T})$ of finitely-generated free models of $\mathcal{T}$ (in Sets). Therefore, Lawvere duality and the central duality result of this thesis yield that

$$\mathbb{C} \cong \mathcal{B}^{op} \simeq \text{Mod}_{\text{fgf}}(\mathcal{T})^{op} \simeq \mathcal{C}_{\mathcal{T}},$$

i.e. that the cube category $\mathbb{C}$ is equivalent to the syntactic category $\mathcal{C}_{\mathcal{T}}$ of the algebraic theory $\mathcal{T}$ with just two constants and no equations.

Our central result that $\mathbb{C}^{op} \cong \mathcal{B}$ will allow us to prove a classifying property of the cube category $\mathbb{C}$, namely that it is the free finite product category on an interval, where an interval in a category $\mathcal{D}$ with terminal object 1 is simply a triple (X, a, b) consisting in an object $X \in \mathcal{D}$, together with two points $a, b : 1 \to X$. We then consider the category $\mathcal{B}_w$ of finite *weakly* bipointed sets, whose objects are finite sets equipped with two possibly equal basepoints, and show that this category also has the classifying property of being the dual of the free finite-limit category on an interval.

We now outline the structure of this thesis by providing a brief summary of each subsequent chapter:

In Chapter 2: Cubes and Bipointed Sets, we present the classical cube category and some possible extensions of it, before presenting our version of an expanded *Cartesian* cube category $\mathbb{C}$ that will be considered throughout this thesis. We then describe some important properties of our cube category $\mathbb{C}$, including the fact that each map has an essentially unique factorization as a composite of basic maps. We also present the category $\mathcal{B}$ of finite strictly bipointed sets, isolate certain basic maps of this category, and then prove that (like $\mathbb{C}$) every map of this category has an essentially unique factorization as a composite of such basic maps.

In Chapter 3: The Duality of $\mathbb{C}$ and $\mathcal{B}$, we first show that the two-element set with suitable structures acts as a dualizing object for a new duality between the categories $\mathbb{C}$ and $\mathcal{B}$, which will be prove to be useful in defining the functors that will witness the duality between these two categories. We then define these functors and prove that they are mutually inverse, thereby establishing the promised duality proof between these two categories. We also present an alternative proof of the duality between $\mathbb{C}$ and $\mathcal{B}$ that makes essential use of Lawvere duality.

Finally, in Chapter 4: Classifying Properties of Cubes and Bipointed Sets, we prove (via the established duality between $\mathcal{B}$ and $\mathbb{C}$) that $\mathbb{C}$ is the free finite-product category on an interval. Then, we introduce the category $\mathcal{B}_w$ of finite *weakly* bipointed sets, and show that this category is dual to the free finite-*limit* category on an interval.

Chapter 2

Cubes and Bipointed Sets

In this chapter we will first review the definition of the classical cube category, and then describe a few ways by which one might extend the cube category. We then define our expanded *Cartesian* version $\mathbb{C}$ of the cube category, and show (in Proposition 1.4) that every map in $\mathbb{C}$ has an essentially unique factorization as a composite of cubical basic maps. Next, we will define the category $\mathcal{B}$ of finite strictly bipointed sets, isolate some basic maps of this category, and then (in Proposition 2.8) that every map in $\mathcal{B}$ likewise has an essentially unique factorization as a composite of these basic maps.

2.1 The Cartesian Cube Category

Various cube categories have been presented in the literature. We immediately define the classical or restricted cube category (see e.g. [6]):

Definition 2.1. The classical cube category Cube is the subcategory of Sets defined as follows:

- *Objects (n-cubes)*: All sets of the form I^n for $n \geq 0$, where $I \doteq \{0, 1\}$. Explicitly, we have $I^0 = \{*\}$, and for any $n \geq 1$:

$$I^n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1\} \text{ for all } 1 \leq i \leq n\}.$$

- *Maps*: The identities, and any function $f : I^m \to I^p$ between cubes that can be obtained by (set-theoretic) composition from (the identities and) the following basic maps:

Degeneracy maps: for any $1 \leq i \leq n + 1$, a map $\varepsilon_i : I^{n+1} \to I^n$ defined as follows:

$$(x_1, \ldots, x_i, \ldots, x_{n+1}) \mapsto (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})$$
(i.e. the i^{th} coordinate is deleted)

Face maps: for any $1 \leq i \leq n+1$ and $\alpha \in \{0,1\}$, a map $\phi_i^\alpha : I^n \to I^{n+1}$ defined as follows:

$$(x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, \alpha, x_i, \dots, x_n)$$
(i.e. α is inserted into the i^{th} position)

Intuitively, a degeneracy map can be thought of as flattening an $(n+1)$-cube along a specified dimension, while a face map can be thought of as mapping an n-cube to some n-dimensional face of the $(n+1)$-cube. For example, if we regard the 1-cube $I = \{0,1\}$ as a line and the 2-cube $I^2 = \{(0,0),(0,1),(1,0),(1,1)\}$ as a square, then we can view the degeneracy map $\varepsilon_1 : I^2 \to I$ as flattening the square to the line along the x-dimension, and we can view the face map $\phi_1^0 : I \to I^2$ as mapping the line to the left-hand face of the square.

To the classical cube category, one may add further kinds of basic maps. For example, one may add *diagonal* maps, described as follows:

Diagonal maps: for any $1 \leq i \leq n$, a map $\delta_i : I^n \to I^{n+1}$ defined as follows:

$$(x_1, \dots, x_i, \dots, x_n) \mapsto (x_1, \dots, x_i, x_i, \dots, x_n)$$

So a diagonal map duplicates the i^{th} coordinate of a binary n-tuple. For example, we can view the diagonal map $\delta_1 : I \to I^2$ as mapping the line to the increasing diagonal (consisting of the pairs $(0,0)$, $(1,1)$) of the square.

Additionally, one may add permutation maps to the classical cube category, described as follows:

Permutation maps: for any $n \geq 0$ and permutation σ of $\{1, \dots, n\}$, a map $\sigma : I^n \to I^n$ in $\mathbb{C}$ defined as follows:

$$(x_1, \dots, x_n) \mapsto (x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

So a permutation map rearranges the coordinates of a binary n-tuple according to the permutation of the set $\{1, \dots, n\}$. For example, the binary swap σ of $\{1,2\}$ induces the permutation map $\sigma : I^2 \to I^2$ that just exchanges the lower-right and upper-left vertices of the square, i.e. that exchanges $(1,0)$ and $(0,1)$. It is worth noting that any permutation map $\sigma : I^n \to I^n$ can be obtained as a composite of permutation maps $\sigma' : I^n \to I^n$ induced by mere binary swaps σ' of $\{1, \dots, n\}$, i.e. permutations σ' such that for some $1 \leq i < n$,

$$\sigma'(i) = i+1 \text{ and } \sigma'(i+1) = i,$$

while for any $1 \leq m < i$,

$$\sigma'(m) = m,$$

and for any $1 < i+1 < m' \leq n$,

$$\sigma'(m') = m'.$$

In this thesis, we will define the Cartesian cube category $\mathbb{C}$ to be the classical cube category with faces and degeneracies, together with (1) diagonals, and (2) permutations. Therefore, we give the following definition:

Definition 2.2. The *Cartesian cube category* $\mathbb{C}$ is the subcategory of Sets defined as follows:

- *Objects (n-cubes)*: All sets of the form I^n for $n \geq 0$, where $I \doteq \{0, 1\}$. Explicitly, we have $I^0 = \{*\}$, and for any $n \geq 1$:

$$I^n = \{(x_1, \ldots, x_n) : x_i \in \{0, 1\} \text{ for all } 1 \leq i \leq n\}.$$

- *Maps*: All functions $f : I^m \to I^p$ between cubes that can be obtained by (set-theoretic) composition from the following basic maps:

Degeneracy maps: for any $1 \leq i \leq n+1$, a map $\varepsilon_i : I^{n+1} \to I^n$ defined as follows:

$$(x_1, \ldots, x_i, \ldots, x_{n+1}) \mapsto (x_1, \ldots, \hat{x_i}, \ldots, x_{n+1})$$

(i.e. the i^{th} coordinate is deleted)

Face maps: for any $1 \leq i \leq n+1$ and $\alpha \in \{0, 1\}$, a map $\phi_i^\alpha : I^n \to I^{n+1}$ defined as follows:

$$(x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_1, \ldots, \alpha, x_i, \ldots, x_n)$$

(i.e. α is inserted into the i^{th} position)

Diagonal maps: for any $1 \leq i \leq n$, a map $\delta_i : I^n \to I^{n+1}$ defined as follows:

$$(x_1, \ldots, x_i, \ldots, x_n) \mapsto (x_1, \ldots, x_i, x_i, \ldots, x_n)$$

Permutation maps: for any $n \geq 0$ and permutation σ of $\{1, \ldots, n\}$, a map $\sigma : I^n \to I^n$ in $\mathbb{C}$ defined as follows:

$$(x_1, \ldots, x_n) \mapsto (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}).$$

In this definition, we note that the identity maps may be obtained as permutation maps. Now, excluding the permutation maps, the basic maps of the Cartesian cube category $\mathbb{C}$ enjoy the following relations among each other, which may be proven by straightforward calculations:

Proposition 2.3. *The following relations hold between the degeneracy, face, and diagonal maps in $\mathbb{C}$:*

1. *The following relations hold between the face and degeneracy maps in $\mathbb{C}$:*

- *If* $j \leq i$, *then*

$$\phi_j^\beta \circ \phi_{i+1}^\alpha = \phi_{i+2}^\alpha \circ \phi_j^\beta$$

and

$$\varepsilon_i \circ \varepsilon_j = \varepsilon_j \circ \varepsilon_{i+1}.$$

- *If* $j < i$, *then*

$$\varepsilon_j \circ \phi_i^\alpha = \phi_{i-1}^\alpha \circ \varepsilon_j.$$

- *If* $j = i$, *then*

$$\varepsilon_j \circ \phi_i^\alpha = id.$$

- *If* $j > i$, *then*

$$\varepsilon_j \circ \phi_i^\alpha = \phi_i^\alpha \circ \varepsilon_{j-1}.$$

2. *The following relations hold between the face and diagonal maps in* $\mathbb{C}$*:*

- *If* $j \leq i$, *then*

$$\delta_j \circ \delta_i = \delta_{i+1} \circ \delta_j.$$

- *If* $j < i$, *then*

$$\delta_j \circ \phi_i^\alpha = \phi_{i+1}^\alpha \circ \delta_j.$$

- *If* $j = i$, *then*

$$\delta_i \circ \phi_i^\alpha = \phi_i^\alpha \circ \phi_i^\alpha.$$

- *If* $j > i$, *then*

$$\delta_j \circ \phi_i^\alpha = \phi_i^\alpha \circ \delta_{j-1}.$$

3. *The following relations hold between the degeneracy and diagonal maps in* $\mathbb{C}$*:*

- *If* $j < i - 1$, *then*

$$\varepsilon_i \circ \delta_j = \delta_j \circ \varepsilon_{i-1}.$$

- *If* $j = i - 1$ *or* $j = i$, *then*

$$\varepsilon_i \circ \delta_j = id.$$

- *If* $j > i$, *then*

$$\varepsilon_i \circ \delta_j = \delta_{j-1} \circ \varepsilon_i.$$

Now we consider the relations between the permutation maps and the other basic maps of the category $\mathbb{C}$ in the following proposition:

Proposition 2.4. *The following relations hold between the permutation maps and the other basic maps in* $\mathbb{C}$*:*

1. *For any permutation map σ of I^{n+1}, we have*

$$\sigma \circ \phi_i^\alpha = \phi_{\sigma(i)}^\alpha \circ \sigma'$$

for a unique permutation map σ' of I^n.

2. *For any permutation map σ of I^{n+1}, we have*

$$\varepsilon_i \circ \sigma = \sigma' \circ \varepsilon_{\sigma^{-1}(i)}$$

for a unique permutation map σ' of I^n.

3. *For any permutation map σ of I^n, we have*

$$\delta_i \circ \sigma = \sigma' \circ \delta_{\sigma^{-1}(i)}$$

for some permutation map σ' of I^{n+1} that need not be unique. However, if we have

$$\delta_i \circ \sigma = \sigma' \circ \delta_{\sigma^{-1}(i)} = \sigma'' \circ \delta_{\sigma^{-1}(i)}$$

for permutation maps σ', σ'' of I^{n+1}, then σ' and σ'' can only differ on $\sigma^{-1}(i)$ and $\sigma^{-1}(i) + 1$.

Proof. The first claim holds because the existence of σ' is fairly obvious, and its uniqueness follows because the face maps are injective and hence monic (as maps in Sets).

For the second claim, it suffices to consider mere binary swaps, since every permutation map of $\{1, \ldots, n+1\}$ can be obtained as a composite of binary swaps. So let σ be a binary swap of $\{1, \ldots, n+1\}$, where we must show that there is a unique binary swap σ' of $\{1, \ldots, n\}$, i.e. of I^n, such that

$$\varepsilon_i \circ \sigma = \sigma' \circ \varepsilon_{\sigma^{-1}(i)}.$$

Before we prove the existence of σ', the uniqueness of σ' follows because the degeneracy maps are surjective and hence epic (as maps in Sets). To prove the existence of σ', we now consider all possible forms of the binary swap σ as follows:

- First suppose that $\sigma(i) = n+1$, which implies both that i is swapped (because $1 \leq i \leq n$) and that $i = n$: then we show that σ' is the identity permutation, i.e. we show that

  $$\varepsilon_n \circ \sigma = \varepsilon_{\sigma^{-1}(n)} = \varepsilon_{n+1}.$$

  So let $(x_1, \ldots, x_{n+1}) \in I^{n+1}$; then we have:

  $$\begin{aligned} \varepsilon_n(\sigma(x_1, \ldots, x_{n+1})) &= \varepsilon_n(x_1, \ldots, x_{n+1}, x_n) \\ &= (x_1, \ldots, x_n) \\ &= \varepsilon_{n+1}(x_1, \ldots, x_{n+1}), \end{aligned}$$

  as desired.

- Now suppose that $\sigma(i) < n+1$, so that there are two further sub-cases: either $\sigma(i) = i$ (i.e. i is not swapped), or i is swapped:

  - First suppose that $\sigma(i) = i$, i.e. that i is not swapped; then the swapping occurs either before or after i.

    First suppose that the swapping occurs before i, i.e. that there is some $2 \leq j < i \leq n$ such that $\sigma(j) = j-1$ and $\sigma(j-1) = j$. Then we show that $\sigma' = \sigma \restriction \{1, \ldots, n\}$, which is clearly a binary swap. To show that

    $$\varepsilon_i \circ \sigma = \sigma' \circ \varepsilon_{\sigma^{-1}(i)} = \sigma' \circ \varepsilon_i,$$

    let $(x_1, \ldots, x_{j-1}, x_j, \ldots, x_i, \ldots, x_{n+1}) \in I^{n+1}$: then we have

    $$\begin{aligned} \varepsilon_i(\sigma(x_1, \ldots, x_{j-1}, x_j, \ldots, x_i, \ldots, x_{n+1})) &= \varepsilon_i(x_1, \ldots, x_j, x_{j-1}, \ldots, x_i, \ldots, x_{n+1}) \\ &= (x_1, \ldots, x_j, x_{j-1}, \ldots, \hat{x}_i, \ldots, x_{n+1}) \\ &= \sigma'(x_1, \ldots, x_{j-1}, x_j, \ldots, \hat{x}_i, \ldots, x_{n+1}) \\ &= \sigma'(\varepsilon_i(x_1, \ldots, x_{j-1}, x_j, \ldots, x_i, \ldots, x_{n+1})), \end{aligned}$$

    as desired.

    Now suppose that the swapping occurs after i, i.e. that there is some $1 \leq i < j \leq n$ such that $\sigma(j) = j+1$ and $\sigma(j+1) = j$. If $j < n$, then we again let $\sigma' = \sigma \restriction \{1, \ldots, n\}$, and the proof that

    $$\varepsilon_i \circ \sigma = \sigma' \circ \varepsilon_{\sigma^{-1}(i)} = \sigma' \circ \varepsilon_i$$

    proceeds almost exactly as in the preceding paragraph. So suppose instead that $j = n$: then we let σ' be the binary swap of $\{1, \ldots, n\}$ that swaps n and $n-1$. To show that

    $$\varepsilon_i \circ \sigma = \sigma' \circ \varepsilon_{\sigma^{-1}(i)} = \sigma' \circ \varepsilon_i,$$

    let $(x_1, \ldots, x_i, \ldots, x_{n+1}) \in I^{n+1}$: then we have

    $$\begin{aligned} \varepsilon_i(\sigma(x_1, \ldots, x_i, \ldots, x_{n+1})) &= \varepsilon_i(x_1, \ldots, x_i, \ldots, x_{n+1}, x_n) \\ &= (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}, x_n) \\ &= \sigma'(x_1, \ldots, \hat{x}_i, \ldots, x_n, x_{n+1}) \\ &= \sigma'(\varepsilon_i(x_1, \ldots, x_i, \ldots, x_{n+1})), \end{aligned}$$

    as desired.

  - Now suppose that i itself is swapped, so that either $\sigma(i) = i-1$ or $\sigma(i) = i+1$. Since the proofs in either case are essentially the same, we show only the proof in the first case. So assume that $\sigma(i) = i-1$, which implies that $1 < i \leq n$. Then we show that σ' is just the identity permutation of $\{1, \ldots, n\}$. To show that

$$\varepsilon_i \circ \sigma = \sigma' \circ \varepsilon_{\sigma^{-1}(i)} = \varepsilon_{i-1},$$

let $(x_1, \ldots, x_{i-1}, x_i, \ldots, x_{n+1}) \in I^{n+1}$: then we have

$$\begin{aligned}\varepsilon_i(\sigma(x_1, \ldots, x_{i-1}, x_i, \ldots, x_{n+1})) &= \varepsilon_i(x_1, \ldots, x_i, x_{i-1}, \ldots, x_{n+1}) \\ &= (x_1, \ldots, \hat{x_{i-1}}, x_i, \ldots, x_{n+1}) \\ &= \varepsilon_{i-1}(x_1, \ldots, x_{i-1}, x_i, \ldots, x_{n+1}),\end{aligned}$$

as desired.

This completes the second case, and so the second claim of the proposition is proved.

For the third claim, the existence of at least one permutation map σ' is fairly obvious. Now we must show that if we have

$$\delta_i \circ \sigma = \sigma' \circ \delta_{\sigma^{-1}(i)} = \sigma'' \circ \delta_{\sigma^{-1}(i)}$$

for some permutation maps σ', σ'' of I^{n+1}, then σ' and σ'' can only differ on $\sigma^{-1}(i)$ and $\sigma^{-1}(i) + 1$ (and they cannot differ on just one of these coordinates, because otherwise one map would not be bijective). So we want to show that if $\sigma' \neq \sigma''$, then both

$$\sigma'(\sigma^{-1}(i)) \neq \sigma''(\sigma^{-1}(i))$$

and

$$\sigma'(\sigma^{-1}(i) + 1) \neq \sigma''(\sigma^{-1}(i) + 1),$$

and moreover there is no $1 \leq j \leq n+1$ such that $j \neq \sigma^{-1}(i)$ and $j \neq \sigma^{-1}(i) + 1$ but $\sigma'(j) \neq \sigma''(j)$, given that

$$\delta_i \circ \sigma = \sigma' \circ \delta_{\sigma^{-1}(i)} = \sigma'' \circ \delta_{\sigma^{-1}(i)}.$$

So assume that $\sigma' \neq \sigma''$. First we show that there is no $1 \leq j \leq n+1$ such that $j \neq \sigma^{-1}(i)$ and $j \neq \sigma^{-1}(i) + 1$ but $\sigma'(j) \neq \sigma''(j)$. So suppose that there is in fact some $1 \leq j \leq n+1$ such that $j \neq \sigma^{-1}(i)$ and $j \neq \sigma^{-1}(i) + 1$ but $\sigma'(j) \neq \sigma''(j)$. But then we cannot have $\sigma' \circ \delta_{\sigma^{-1}(i)} = \sigma'' \circ \delta_{\sigma^{-1}(i)}$, contrary to assumption, because σ' and σ'' will be distinctly permuting a coordinate that is unaffected by $\delta_{\sigma^{-1}(i)}$.

Now we show that

$$\sigma'(\sigma^{-1}(i)) \neq \sigma''(\sigma^{-1}(i)).$$

So suppose that in fact

$$\sigma'(\sigma^{-1}(i)) = \sigma''(\sigma^{-1}(i)).$$

Then since $\sigma' \neq \sigma''$ and σ' and σ'' are equal on all coordinates outside of $\sigma^{-1}(i)$ and $\sigma^{-1}(i) + 1$ (by what was just shown in the previous paragraph), it must be that

$$\sigma'(\sigma^{-1}(i) + 1) \neq \sigma''(\sigma^{-1}(i) + 1).$$

So σ' and σ'' agree on all coordinates but one, which implies that at least one of σ' and σ'' is not bijective, which is impossible. Therefore, we have

$$\sigma'(\sigma^{-1}(i)) \neq \sigma''(\sigma^{-1}(i)),$$

and parallel reasoning also shows that

$$\sigma'(\sigma^{-1}(i)+1) \neq \sigma''(\sigma^{-1}(i)+1).$$

Thus, we have shown that if

$$\delta_i \circ \sigma = \sigma' \circ \delta_{\sigma^{-1}(i)} = \sigma'' \circ \delta_{\sigma^{-1}(i)},$$

then σ' and σ'' can differ on only $\sigma^{-1}(i)$ and $\sigma^{-1}(i)+1$, so that the permutation map σ is unique up to its action on the coordinates doubled by $\delta_{\sigma^{-1}(i)}$. This completes the proof of the third claim, and hence of Proposition 2.4.

□

Propositions 2.3 and 2.4 have been leading up to the following proposition stating that every map in the category $\mathbb{C}$ has an essentially unique factorization as a composite of basic maps.

Proposition 2.5. *Any map $\lambda : I^n \to I^m$ in $\mathbb{C}$ has a factorization of the following form:*

$$\lambda = \phi_{q_j}^{\alpha_j} \circ \ldots \circ \phi_{q_1}^{\alpha_1} \circ \sigma \circ \delta_{i_k} \circ \ldots \circ \delta_{i_1} \circ \varepsilon_{r_s} \circ \ldots \circ \varepsilon_{r_1}$$
$$: I^n \to \ldots \to I^{n-s} \to \ldots \to I^{n-s+k} \to \ldots \to I^{n-s+k+l} = I^m,$$

where $\sigma : I^{n-s+k} \to I^{n-s+k}$, and where $r_s < \ldots < r_1, q_1 < \ldots < q_j$, $i_1 < \ldots < i_k, \alpha_1, \ldots, \alpha_j \in \{0,1\}$, as in the following diagram:

$$\begin{array}{ccc} I^n & \xrightarrow{\lambda} & I^m \\ {\scriptstyle\varepsilon}\downarrow & & \uparrow{\scriptstyle\phi} \\ I^{n-s} & \xrightarrow{\sigma \circ \delta} & I^{n-s+k} \end{array}$$

This factorization is essentially unique *in the following sense: given two factorizations*

$$\lambda = \phi_{q_j}^{\alpha_j} \circ \ldots \circ \phi_{q_1}^{\alpha_1} \circ \sigma \circ \delta_{i_k} \circ \ldots \circ \delta_{i_1} \circ \varepsilon_{r_s} \circ \ldots \circ \varepsilon_{r_1}$$

and

$$\lambda = \phi_{q'_{j'}}^{\alpha'_{j'}} \circ \ldots \circ \phi_{q'_1}^{\alpha'_1} \circ \sigma' \circ \delta_{i'_{k'}} \circ \ldots \circ \delta_{i'_1} \circ \varepsilon_{r'_{s'}} \circ \ldots \circ \varepsilon_{r'_1}$$

such that $r_s < \ldots < r_1, q_1 < \ldots < q_j, i_1 < \ldots < i_k, \alpha_1, \ldots, \alpha_j \in \{0,1\}$ and $r'_{s'} < \ldots < r'_1, q'_1 < \ldots < q'_{j'}, i'_1 < \ldots < i'_{k'}, \alpha'_1, \ldots, \alpha'_{j'} \in \{0,1\}$, the following equalities hold:

- *$s = s'$ and $r_p = r'_p$ for every $1 \leq p \leq s$.*
- *$j = j'$ and $q_p = q'_p$ and $\alpha_p = \alpha'_p$ for every $1 \leq p \leq j$.*

- *$k = k'$ and $i_p = i'_p$ for every $1 \leq p \leq k$.*

Moreover, if $\sigma \neq \sigma'$, then for any $i \in \{1, \ldots, n-s+k\}$ such that $\sigma(i) \neq \sigma'(i)$, there is some $1 \leq p \leq k$ such that $i = i_p$ or $i = i_p + 1$.

Proof. The existence of the desired factorization follows from the relations among the basic maps of $\mathbb{C}$ presented in Propositions 2.3 and 2.4, given that every map in $\mathbb{C}$ is by definition a composite of some basic maps (but perhaps not of the form required by this proposition), together with the obvious fact that a composite of permutation maps can be regarded as a single permutation map.

To prove the essential uniqueness of the factorization, suppose that we have factorizations

$$\lambda = \phi_{q_j}^{\alpha_j} \circ \ldots \circ \phi_{q_1}^{\alpha_1} \circ \sigma \circ \delta_{i_k} \circ \ldots \circ \delta_{i_1} \circ \varepsilon_{r_s} \circ \ldots \circ \varepsilon_{r_1}$$

and

$$\lambda = \phi_{q'_{j'}}^{\alpha'_{j'}} \circ \ldots \circ \phi_{q'_1}^{\alpha'_1} \circ \sigma' \circ \delta_{i'_{k'}} \circ \ldots \circ \delta_{i'_1} \circ \varepsilon_{r'_{s'}} \circ \ldots \circ \varepsilon_{r'_1}$$

of the desired forms. For now, let us abbreviate the factorizations as

$$\lambda = \phi \circ \sigma \circ \delta \circ \varepsilon = \phi' \circ \sigma' \circ \delta' \circ \varepsilon',$$

where ϕ, ϕ' denote the corresponding composites of face maps, and so on. Now, since the face maps, permutation maps, and diagonal maps are all injective, while the degeneracy maps are surjective, it follows that λ, as a function in the category Sets, has two surjective-injective factorizations: one factorization with surjective part ε and injective part $\phi \circ \sigma \circ \delta$, and another factorization with surjective part ε' and injective part $\phi' \circ \sigma' \circ \delta'$. Then, since surjective-injective factorizations of set functions are unique up to isomorphism, and since isomorphic objects in $\mathbb{C}$ are equal, it follows that ε and ε' have isomorphic and hence equal codomains, and that $\phi \circ \sigma \circ \delta$ and $\phi' \circ \sigma' \circ \delta'$ have isomorphic and hence equal domains, which entails that (extensionally speaking)

$$\varepsilon = \varepsilon'$$

and

$$\phi \circ \sigma \circ \delta = \phi' \circ \sigma' \circ \delta'.$$

Given that $\varepsilon = \varepsilon'$, we now show that these two composites of degeneracy maps have identical factorizations, i.e. we show that $s = s'$ and $r_p = r'_p$ for every $1 \leq p \leq s$. First, the fact that $\varepsilon = \varepsilon'$ entails that $s = s'$, as desired. Now we show by induction on $1 \leq p \leq s$ that $r_p = r'_p$. So let $1 \leq p \leq s$ be such that for any $u < p$, we have $r_u = r'_u$, where we show that $r_p = r'_p$. Well, if $r_p \neq r'_p$, then suppose without loss of generality that $r_p < r'_p$. Then since $r_s < \ldots < r_p$, we have $r_s < \ldots < r_p < r'_p$, and so the coordinate deleted by $\varepsilon_{r'_p}$ will never be deleted by $e_{r_s} \circ \ldots \circ \varepsilon_{r_p}$, so that $\varepsilon \neq \varepsilon'$, which is impossible. Therefore, we have by mathematical induction that $r_p = r'_p$ for any $1 \leq p \leq s$, whereby ε and ε' have identical factorizations, as desired.

Since ϕ and ϕ' are completely determined by the image of λ, it follows that ϕ and ϕ' must have identical factorizations, so that in particular $\phi = \phi'$. Then since ϕ is monic and $\phi \circ \sigma \circ \delta = \phi \circ \sigma' \circ \delta'$, it follows that $\sigma \circ \delta = \sigma' \circ \delta'$.

To show that δ and δ' have identical factorizations, first note that since $\sigma \circ \delta = \sigma' \circ \delta'$, it follows that $k = k'$. Now we show by induction on $1 \leq p \leq k$ that $i_p = i'_p$. So let $1 \leq p \leq k$ be such that for any $u < p$, we have $i_u = i'_u$, where we show that $i_p = i'_p$. Well, if $i_p \neq i'_p$, suppose without loss of generality that $i_p < i'_p$. But then since $i'_p < \ldots < i'_k$, we have $i_p < i'_p < \ldots < i'_k$, so that δ' will never duplicate the i_p^{th} coordinate. However, this entails that $\sigma \circ \delta \neq \sigma' \circ \delta'$, which is impossible. Therefore, we have that $i_p = i'_p$ and by induction that the factorizations of δ and δ' are identical, so that in particular we have $\sigma \circ \delta = \sigma' \circ \delta$. However, Proposition 2.4(3) entails that we may have $\sigma \neq \sigma'$, although Proposition 2.4 entails that if $\sigma \neq \sigma'$, then for any $i \in \{1, \ldots, n - s + k\}$ such that $\sigma(i) \neq \sigma'(i)$, there is some $1 \leq p \leq k$ such that $i = i_p$ or $i = i_p + 1$. This completes the proof that the factorization of λ is essentially unique.

□

This concludes Section 2.1, in which we have defined the Cartesian cube category $\mathbb{C}$ whose objects are the n-cubes and whose maps are all and only those set functions generated by the degeneracies, faces, diagonals, and permutations (binary swaps). Also, as we have just seen, each map in the category $\mathbb{C}$ has an essentially unique factorization consisting of degeneracies followed by diagonals followed by a permutation followed by faces.

2.2 Finite Strictly Bipointed Sets

Now that we have defined the Cartesian cube category $\mathbb{C}$, we proceed to define the category $\mathcal{B}$ of finite strictly bipointed sets, to which $\mathbb{C}$ will be dual (as we will prove in Chapter 3). Without further ado, the category $\mathcal{B}$ is defined as follows:

Definition 2.6. The category $\mathcal{B}$ of *finite strictly bipointed sets* is defined as follows:

- *Objects*: For any $n \geq 0$, the finite strictly bipointed set

$$[n] \doteq (\{1, \ldots, n, -, +\}, -, +).$$

- *Maps*: A map $f : [n] \to [m]$ is any set function

$$f : \{1, \ldots, n, -, +\} \to \{1, \ldots, m, -, +\}$$

  that preserves the basepoints, i.e.

$$f(-) = -,$$

$$f(+) = +.$$

Composition and identities are inherited from Sets.

So an object $[n]$ of $\mathcal{B}$ is a set-theoretic triple with underlying set $\{1, \ldots, n, -, +\}$ together with two basepoints $-$ and $+$, where we usually write $[n]$ as just

$$[n] = \{1, \ldots, n, -, +\},$$

with the obvious basepoints $-, +$. In particular, we have

$$[0] = \{-, +\},$$

the initial object of $\mathcal{B}$. It turns out that we can isolate four different kinds of basic maps of $\mathcal{B}$, from which we can prove a factorization theorem for the maps of $\mathcal{B}$ parallel to the factorization result for the maps of $\mathbb{C}$ (Proposition 2.5). We introduce these basic maps of $\mathcal{B}$ as follows:

Definition 2.7. We define the following four kinds of basic maps of $\mathcal{B}$:

1. *Co-degeneracy maps*: For any $1 \leq i \leq n+1$, a map

   $$e_i : [n] \to [n+1]$$

   defined as follows:

   - e_i preserves the basepoints (as required to be a map in $\mathcal{B}$).
   - For any $1 \leq j < i$,
     $$e_i(j) = j.$$
   - For any $i \leq k \leq n$,
     $$e_i(k) = k + 1.$$

2. *Co-face maps*: For any $1 \leq i \leq n+1$ and $\alpha \in \{-, +\}$, a map

   $$f_i^\alpha : [n+1] \to [n]$$

   defined as follows:

   - f_i^α preserves the basepoints.
   - For any $1 \leq j < i$,
     $$f_i^\alpha(j) = j.$$
   - $f_i^\alpha(i) = \alpha$.
   - For any $i < k \leq n+1$,
     $$f_i^\alpha(k) = k - 1.$$

3. *Co-diagonal maps*: For any $1 \leq i \leq n$, a map

   $$d_i : [n+1] \to [n]$$

   defined as follows:

- d_i preserves the basepoints.
- For any $1 \leq j \leq i$,
$$d_i(j) = j.$$
- $d_i(i+1) = i$.
- For any $i + 1 < k \leq n + 1$,
$$d_i(k) = k - 1.$$

4. *Co-permutation maps*: For any $n \geq 0$ and permutation s of $\{1, \ldots, n\}$, a permutation map
$$s : [n] \to [n]$$
that preserves the basepoints.

Intuitively speaking, a co-degeneracy map omits a specified coordinate from its image, a co-face map sends a specified coordinate in its domain (other than the basepoints) to a basepoint, a co-diagonal map sends two specified coordinates in its domain to the same coordinate in its image, and the effect of a co-permutation map is obvious.

We can now prove a factorization result for $\mathcal{B}$ in some sense dual to the corresponding result of Proposition 2.5, namely that every map of $\mathcal{B}$ can be essentially uniquely factorized as a certain composite of the basic maps just defined:

Proposition 2.8. *Any map $g : [n] \to [m]$ in $\mathcal{B}$ has a factorization of the following form:*

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i_1} \circ \ldots \circ d_{i_k} \circ s \circ f_{q_1}^{\alpha_1} \circ \ldots \circ f_{q_j}^{\alpha_j}$$

$$: [n] \to \ldots \to [n-j] \to \ldots \to [n-j-k] \to \ldots \to [n-j-k+s] = [m],$$

where $s : [n-j] \to [n-j]$, and where $r_s < \ldots < r_1, q_1 < \ldots < q_j$, $i_1 < \ldots i_k, \alpha_1, \ldots, \alpha_j \in \{-, +\}$, as in the following diagram:

$$\begin{array}{ccc} [n] & \xrightarrow{\quad g \quad} & [m] \\ {\scriptstyle f}\big\downarrow & & \big\uparrow{\scriptstyle e} \\ [n-j] & \xrightarrow{d \circ s} & [n-j-k] \end{array}$$

This factorization is essentially unique *in the following sense: given two factorizations*

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i_1} \circ \ldots \circ d_{i_k} \circ s \circ f_{q_1}^{\alpha_1} \circ \ldots \circ f_{q_j}^{\alpha_j}$$

and

$$g = e_{r'_1} \circ \ldots \circ e_{r'_{s'}} \circ d_{i'_1} \circ \ldots \circ d_{i'_{k'}} \circ s' \circ f_{q'_1}^{\alpha'_1} \circ \ldots \circ f_{q'_{j'}}^{\alpha'_{j'}}$$

such that $r_s < \ldots < r_1, q_1 < \ldots < q_j, i_1 < \ldots i_k, \alpha_1, \ldots, \alpha_j \in \{-, +\}$ and $r'_{s'} < \ldots < r'_1, q'_1 < \ldots < q'_{j'}, i'_1 < \ldots i'_{k'}, \alpha'_1, \ldots, \alpha'_{j'} \in \{-, +\}$, the following equalities hold:

- *$s = s'$ and $r_p = r'_p$ for every $1 \leq p \leq s$.*
- *$j = j'$ and $q_p = q'_p$ and $\alpha_p = \alpha'_p$ for every $1 \leq p \leq j$.*
- *$k = k'$ and $i_p = i'_p$ for every $1 \leq p \leq k$.*

Moreover, if $s \neq s'$, then for any $i \in \{1, \ldots, n-j\}$ such that $s(i) \neq s'(i)$, there is some $1 \leq p \leq k$ such that $i = i_p$ or $i = i_p + 1$.

Proof. (Sketch) To prove the existence of such a factorization for g, we first let $q_1 < \ldots < q_j \in \{1, \ldots, n\}$ be all and only those natural numbers q such that $g(q) \in \{-, +\}$. Then we begin writing a factorization of g as

$$g = \ldots \circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)} : [n] \to [n-j].$$

Next, we work backwards from the end of the desired factorization. So let $r_s < \ldots < r_1 \in \{1, \ldots, m\}$ be all and only those natural numbers such that for any $1 \leq u \leq s$, there is no $r \in \{1, \ldots, n\}$ such that $g(r) = r_u$ (i.e. g does not 'hit' any member of the sequence $r_s < \ldots < r_1$). Then we write the end of the desired factorization of g as follows:

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ \ldots ? \ldots \circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)}.$$

Now, we continue working backwards from the co-degeneracy maps to the co-diagonal maps: let $i_1 < \ldots < i_k \in \{1, \ldots, m\}$ be all and only those natural numbers with respect to which g is not injective, i.e. such that for any $1 \leq v \leq k$, there are at least two $v', v'' \in \{1, \ldots, n\}$ such that $g(v') = g(v'') = i_v$. Consider first i_1: by the preceding paragraph, since i_1 is in the image of g, there is a unique $i^1 \in \{1, \ldots, m-s\}$ such that

$$(e_{r_1} \circ \ldots \circ e_{r_s})(i^1) = i_1$$

(unique, because the co-degeneracy maps are injective). So we continue writing the factorization of g as

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \ldots ? \ldots \circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)}.$$

Now, let $\#i_1 \geq 2$ be the number of $z \in \{1, \ldots, n\}$ such that $g(z) = i_1$. Then we continue writing the factorization of g as

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots ? \ldots \circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)}.$$

Now consider i_2: there will be a unique $i^2 \in \{1, \ldots, m-s+(\#i_1 - 1)\}$ such that

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2})(i^2) = i_2.$$

So we continue writing the factorization of g as

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ d_{i^2} \circ \ldots ? \ldots \circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)}.$$

Again, let $\#i_2 \geq 2$ be the number of $z \in \{1, \ldots, n\}$ such that $g(z) = i_2$. Then we continue writing the factorization of g thus far as

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ d_{i^2} \circ d_{i^2+1} \circ \ldots \circ d_{i^2+\#i_2-2}$$

$$\circ \ldots ? \ldots \circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)}.$$

We now continue writing the factorization of g in this fashion as follows:

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2}$$

$$\circ \ldots ? \ldots \circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)},$$

where for each $1 < v \leq k$, i^v is the unique element of $\{1, \ldots, m - s + (\#i_1 - 1) + \ldots + (\#i_{v-1} - 1)\}$ such that

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^{v-1}} \circ \ldots \circ d_{i^{v-1}+\#i_{v-1}-2})(i^v) = i_v,$$

and $\#i_v \geq 2$ is the number of $z \in \{1, \ldots, n\}$ such that $g(z) = i_z$.

Now we complete the factorization of g by creating a (co-)permutation map $s : [n-j] \to [n-j]$. So consider any

$$w \in \{1, \ldots, n-j\} = \{1, \ldots, m - s + (\#i_1 - 1) + \ldots + (\#i_k - 1)\}.$$

Since the co-face maps are all surjective and injective (disregarding the natural numbers that co-face maps send to basepoints), we have that

$$w = (f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)})(w')$$

for a unique $w' \in \{1, \ldots, n\}/\{q_1, \ldots, q_j\}$. Now, consider $g(w') \in \{1, \ldots, m\}$: if $g(w') \notin \{i_1, \ldots, i_k\}$, then there will be a unique $w'' \in \{1, \ldots, n-j\}$ such that

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2})(w'') = g(w'),$$

so we let $s(w) = w''$. Otherwise, if $g(w') \in \{i_1, \ldots, i_k\}$, suppose $g(w') = i_v$ for some $1 \leq v \leq k$. Then there will be $\#i_v$ elements $z \in \{1, \ldots, n-j\}$ such that

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2})(z) = g(w') = i_v.$$

Now let $\{w_1, \ldots, w_{\#i_v}\}$ be the set of all $z \in \{1, \ldots, n-j\}$, ordered from least to greatest, such that for any $1 \leq t \leq \#i_v$, there is a unique $w'_t \in \{1, \ldots, n\}$ such that

$$(f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)})(w'_t) = w_t$$

and such that $g(w'_t) = i_v$. Now, we let $s(w_t) = i^v + (t-1)$ for each $1 \leq t \leq \#i_v$. This concludes the definition of s, and it is not too hard to show that s is a permutation map in $\mathcal{B}$. So we complete the factorization of g as

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2} \circ s$$

$$\circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)}.$$

Now we actually prove that, in fact,

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2} \circ s$$

$$\circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)},$$

by showing that for any $z \in \{1, \ldots, n\}$, we have

$$g(z) = (e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2} \circ s$$

$$\circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)})(z).$$

First, let $z \in \{q_1, \ldots, q_j\}$, say $z = q_k$, so that $g(z) = g(q_k) \in \{-, +\}$. Then by definition of the co-face maps, we have

$$(f_{q_{k+1}}^{g(q_{k+1})} \circ \ldots \circ f_{q_j}^{g(q_j)})(q_k) = g(q_k),$$

since $q_k < q_{k+1} < \ldots < q_j$, and hence

$$(f_{q_k}^{g(q_k)} \circ f_{q_{k+1}}^{g(q_{k+1})} \circ \ldots \circ f_{q_j}^{g(q_j)})(q_k) = g(q_k) \in \{-, +\}.$$

Then since all of the maps in $\mathcal{B}$ are basepoint-preserving, it follows that

$$g(q_k) = (e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ d_{i^1+1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2} \circ s$$

$$\circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)})(q_k),$$

as desired.

Next, let $w' \in \{1, \ldots, n\}/\{q_1, \ldots, q_j\}$ be such that g is injective with respect to w'. Then we have

$$(f_{q_1}^{g(q_1)} \circ \ldots \circ f^{g(q_j)})(w') = w \in \{1, \ldots, n-j\},$$

and then we have $s(w) = w''$, where we have

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2})(w'') = g(w'),$$

so that

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2} \circ s$$

$$\circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)})(w') = g(w'),$$

as desired.

Lastly, let $w'_t \in \{1, \ldots, n\}/\{q_1, \ldots, q_j\}$ be such that $g(w'_t) = i_v$, where $1 \leq v \leq k$ and $1 \leq t \leq \#i_v$. Then we have

$$(f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)})(w'_t) = w_t \in \{1, \ldots, n-j\},$$

and then we have $s(w_t) = i_{v_t}$, where we then have

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^{v-1}} \circ \ldots \circ d_{i^{v-1}+\#i_{v-1}-2})(i_{v_t}) = i_v = g(w'_t).$$

Thus, we have

$$(e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2} \circ s$$

$$\circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)})(w'_t) = g(w'_t),$$

as desired. This completes the proof that

$$g = e_{r_1} \circ \ldots \circ e_{r_s} \circ d_{i^1} \circ \ldots \circ d_{i^1+\#i_1-2} \circ \ldots \circ d_{i^k} \circ \ldots \circ d_{i^k+\#i_k-2} \circ s$$

$$\circ f_{q_1}^{g(q_1)} \circ \ldots \circ f_{q_j}^{g(q_j)}.$$

The essential uniqueness conditions on the factorization of g may be proven in much the same way as the essential uniqueness conditions were proven in Proposition 2.5. In sketch form, the co-degeneracy maps are completely determined by the points of $[m]$ that g omits from the image, while the co-face maps are completely determined by the points of $[n]$ (other than the basepoints) that g maps to the basepoints of $[m]$. The (co-)permutation map is uniquely determined up to the representatives in $\{1, \ldots, n-j\}$ of the points with respect to which g is not injective. Finally, the co-diagonal maps are completely determined by the points of $[n]$ with respect to which g is not injective.

□

This concludes Section 2.2, in which we have defined the category $\mathcal{B}$ of finite strictly bipointed sets as well as its basic maps, and where each map of this category has an essentially unique factorization as a composite of such basic maps.

Chapter 3

The Duality of $\mathbb{C}$ and $\mathcal{B}$

In this chapter we first present a new Stone-type duality between the Cartesian cube category $\mathbb{C}$ and the category $\mathcal{B}$ of finite strictly bipointed sets, which is mediated by hom-ing into the two-element set. This can be seen as a Stone-type duality in that it is mediated by hom-ing into a two-element set, which can be regarded as an object of both categories. We then use the isomorphisms involved in this duality to define functors $F : \mathbb{C}^{op} \to \mathcal{B}$ and $G : \mathcal{B}^{op} \to \mathbb{C}$, which we will then prove to be mutually inverse, thereby establishing the duality-isomorphism

$$\mathbb{C}^{op} \cong \mathcal{B}$$

between $\mathbb{C}$ and $\mathcal{B}$.

3.1 A New Duality

In this section we present a Stone-type duality between $\mathbb{C}$ and $\mathcal{B}$, which is mediated by hom-ing into a two-element set as dualizing object, which can be viewed as an object of both categories. Specifically, we have the 1-cube $I = \{0, 1\}$ in the Cartesian cube category $\mathbb{C}$, as well as the finite strictly bipointed set $[0] = (\{-, +\}, -, +)$ in the category $\mathcal{B}$. This new Stone-type duality will prove to be very useful in defining the mutually inverse functors that will witness the duality between the categories $\mathbb{C}$ and $\mathcal{B}$. We will first prove that in the category Sets, there (for any $m \geq 0$) is an isomorphism

$$I^m \cong \mathrm{Hom}_{\mathcal{B}}([m], [0]),$$

where I^m is the m-cube of the category $\mathbb{C}$, and $[m]$ is the $(m+2)$-element finite strictly bipointed set of the category $\mathcal{B}$. We will then prove that in the category Sets, there is (for any $m \geq 0$) also a dual isomorphism

$$[m] \cong \mathrm{Hom}_{\mathbb{C}}(I^m, I),$$

where we forget about the basepoints of $[m]$ (in the sense that we will be constructing an isomorphism $\{1, \dots, m, -, +\} \cong \mathrm{Hom}_{\mathbb{C}}(I^m, I)$). Without further ado, here is the proof of the first (more obvious) isomorphism:

Proposition 3.1. *In* Sets*, we have the following isomorphism:*

$$i_m : I^m \cong \mathrm{Hom}_{\mathcal{B}}([m],[0]).$$

Proof. For any $m \geq 0$, we define a bijective function

$$i_m : I^m \to \mathrm{Hom}_{\mathcal{B}}([m],[0])$$

as follows. First let $m = 0$. Then $I^0 = \{*\}$ and $\mathrm{Hom}_{\mathcal{B}}([0],[0]) = \{id_{[0]}\}$, so that

$$i_0 : I^0 \to \mathrm{Hom}_{\mathcal{B}}([0],[0]),$$

is the evident isomorphism.

Now let $m \geq 1$, and consider an arbitrary binary m-tuple $(x_1, \ldots, x_m) \in I^m$, where we define

$$i_m(x_1, \ldots, x_m) : [m] \to [0]$$

in $\mathcal{B}$. First, $i_m(x_1, \ldots, x_m)$ must preserve the basepoints to be a map in $\mathcal{B}$, and if $1 \leq j \leq m$, then we define

$$i_m(x_1, \ldots, x_m)(j) \in \{-,+\}$$

as follows:

$$i_m(x_1, \ldots, x_m)(j) = \begin{cases} - & \text{if } x_j = 0 \\ + & \text{if } x_j = 1. \end{cases}$$

This defines

$$i_m(x_1, \ldots, x_m) : [m] \to [0]$$

as a map in $\mathcal{B}$, for any binary m-tuple $(x_1, \ldots, x_m) \in I^m$.

Now we define $i_m^{-1} : \mathrm{Hom}_{\mathcal{B}}([m],[0]) \to I^m$ and then show that i_m and i_m^{-1} are mutually inverse. So let $h : [m] \to [0]$ be any map in $\mathcal{B}$, where we define

$$i_m^{-1}(h) \in I^m$$

as a binary m-tuple. Where

$$w : \{-,+\} \to \{0,1\}$$

is the function mapping

$$- \mapsto 0$$

$$+ \mapsto 1,$$

we set

$$i_m^{-1}(h) = (w(h(1)), \ldots, w(h(m))) \in I^m.$$

Now we show that i_m and i_m^{-1} are mutually inverse. First let $(x_1, \ldots, x_m) \in I^m$, where we show that

$$i_m^{-1}(i_m(x_1, \ldots, x_m)) = (x_1, \ldots, x_m).$$

To do this, let $1 \leq j \leq m$, where we show that the j^{th} component of $i_m^{-1}(i_m(x_1, \ldots, x_m))$ is x_j.

If $x_j = 0$, then the j^{th} component of $i_m^{-1}(i_m(x_1, \ldots, x_m))$ is

$$w(i_m(x_1, \ldots, x_m)(j)) = w(-) = 0 = x_j,$$

as required.

Similarly, if $x_j = 1$, then the j^{th} component of $i_m^{-1}(i_m(x_1, \ldots, x_m))$ is $1 = x_j$. Thus, we have that

$$i_m^{-1}(i_m(x_1, \ldots, x_m)) = (x_1, \ldots, x_m),$$

as required.

Now let $h : [m] \to [0]$ be a map in $\mathcal{B}$, where we show that

$$i_m(i_m^{-1}(h)) = h.$$

To do this, let $1 \leq j \leq m$, where we show that

$$i_m(i_m^{-1}(h))(j) = h(j).$$

First, we have

$$i_m^{-1}(h) = (w(h(1)), \ldots, w(h(m))).$$

Now suppose that $h(j) = -$, so that $w(h(j)) = 0$. Then we have

$$i_m(i_m^{-1}(h))(j) = i_m(w(h(1)), \ldots, w(h(m)))(j) = - = h(j),$$

since $w(h(j)) = 0$. Similarly, if $h(j) = +$, then we have

$$i_m(i_m^{-1}(h))(j) = + = h(j).$$

Thus, we have

$$i_m(i_m^{-1}(h)) = h,$$

as required. So i_m and i_m^{-1} are mutually inverse, whence we do have

$$I^m \cong \mathrm{Hom}_{\mathcal{B}}([m], [0]),$$

as desired.

□

Here is the proof of the second (less obvious) isomorphism:

Proposition 3.2. *In* Sets*, we have the following isomorphism:*

$$j_m : [m] \cong \mathrm{Hom}_{\mathbb{C}}(I^m, I).$$

Proof. We prove separately for $m = 0$ and $m \geq 1$ that there is a bijective function

$$j_m : \{1, \ldots, m, -, +\} \to \mathrm{Hom}_{\mathbb{C}}(I^m, I).$$

If $m = 0$, then we must show

$$\{-, +\} \cong \mathrm{Hom}_{\mathbb{C}}(\{*\}, I).$$

But there are exactly two maps in $\mathbb{C}$ from $\{*\}$ to I, namely the two face maps ϕ_0^0 and ϕ_0^1. So we have a bijection

$$j_0 : \{-, +\} \cong \mathrm{Hom}_{\mathbb{C}}(\{*\}, I).$$

Now we prove the result for $m \geq 1$. So we want to construct a bijective function

$$j_m : \{1, \ldots, m, -, +\} \to \mathrm{Hom}_{\mathbb{C}}(I^m, I).$$

First, we map the basepoints $-$ and $+$ to the two constant maps from I^m to I, i.e. the maps

$$\phi_0^0 \circ \varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{m-1} \circ \varepsilon_m : I^m \to I$$

and

$$\phi_0^1 \circ \varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{m-1} \circ \varepsilon_m : I^m \to I.$$

Now let $1 \leq k \leq m$, where we map k to the k^{th} projection map from I^m to I. These projection maps are maps in $\mathbb{C}$, which can be seen as follows:

- We map m to
$$\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{m-1} : I^m \to I;$$
- We map 1 to
$$\varepsilon_2 \circ \varepsilon_3 \circ \ldots \circ \varepsilon_{m-1} \circ \varepsilon_m : I^m \to I;$$
- If $1 < k < m$, we map k to
$$\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{k-1} \circ \varepsilon_{k+1} \circ \ldots \circ \varepsilon_{m-1} \circ \varepsilon_m : I^m \to I.$$

So

$$j_m : \{1, \ldots, m, -, +\} \to \mathrm{Hom}_{\mathbb{C}}(I^m, I)$$

is an injective mapping.

To show that this mapping is also surjective, let $\lambda : I^m \to I$ be any map in $\mathbb{C}$, where we reason as follows to show that λ is in the image of j_m. By Proposition 2.5, λ has a factorization in $\mathbb{C}$ of the following form:

$$\lambda = \phi_{q_j}^{\alpha_j} \circ \ldots \circ \phi_{q_1}^{\alpha_1} \circ \sigma \circ \delta_{i_k} \circ \ldots \circ \delta_{i_1} \circ \varepsilon_{r_s} \circ \ldots \circ \varepsilon_{r_1}.$$

Since the codomain of λ is I, it follows that either $j = 1$ or $j = 0$, i.e. that there is at most one face map in the factorization of λ. If $j = 1$, then σ can only be the identity, and moreover $k = 0$ (i.e. there are no diagonal maps in the factorization of λ), and so

$$\lambda = \phi_0^i \circ \varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{m-1} \circ \varepsilon_m$$

(because λ has domain I^m) with $i \in \{0, 1\}$, i.e. λ is a constant map. But both constant maps from I^m to I are in the image of j_m, and so λ is in the image of j_m.

Now suppose that $j = 0$, so that there are no face maps in the factorization of λ. Then since the codomain of λ is I, σ again can only be the identity, and it must be that $k = 0$, i.e. that there are no diagonal maps in the factorization of λ, because a diagonal map cannot have I as codomain. Hence, if $j = 0$, the factorization of λ consists solely of degeneracy maps, and so by the required ordering on the indices of the degeneracy maps in the factorization (see Proposition 2.5), we have that λ is the k^{th} projection map from I^m to I, for some $1 \leq k \leq m$. But these maps are all in the image of j_m, and hence λ is in the image of j_m. So in any case, we have that $\lambda : I^m \to I$ is in the image of j_m, whereby j_m is surjective. Thus,

$$j_m : \{1, \ldots, m, -, +\} \to \mathrm{Hom}_{\mathbb{C}}(I^m, I)$$

is bijective, as desired.

□

In this section we have constructed two isomorphisms in the category Sets that will be used to construct a Stone-type duality given by the two-element set as dualizing object. These isomorphisms will be very important in defining the functors that form the duality between $\mathbb{C}$ and $\mathcal{B}$, as described in the next section.

3.2 The Duality Proof

In this section we will define functors $F : \mathbb{C}^{op} \to \mathcal{B}$ and $G : \mathcal{B}^{op} \to \mathbb{C}$ that we will prove to be mutually inverse, thus establishing the duality of $\mathbb{C}$ and $\mathcal{B}$. First we will define the functor $F : \mathbb{C}^{op} \to \mathcal{B}$. The difficulty in defining F lies in defining F on the maps of $\mathbb{C}$. To do this, we will make use of the second isomorphism

$$j_m : \{1, \ldots, m, -, +\} \to \mathrm{Hom}_{\mathbb{C}}(I^m, I)$$

just defined in Section 3.1, as well as hom-ing into the fixed object I of $\mathbb{C}$. We now present the following definition of $F : \mathbb{C}^{op} \to \mathcal{B}$, which we will need to prove is actually a functor that lands in the category $\mathcal{B}$.

Definition 3.3. We define the functor $F : \mathbb{C}^{op} \to \mathcal{B}$ as follows:

- *Objects*: For any n-cube $I^n \in \mathbb{C}$, we set

$$F(I^n) = [n] \in \mathcal{B}.$$

- *Maps*: For any map $\lambda : I^n \to I^m$ in $\mathbb{C}$, we set

$$F(\lambda) = j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\lambda, I) \circ j_m$$

$$: [m] \to \mathrm{Hom}_{\mathbb{C}}(I^m, I) \to \mathrm{Hom}_{\mathbb{C}}(I^n, I) \to [n],$$

as in the following diagram:

$$\begin{array}{ccc} [m] & \xrightarrow{\quad F(\lambda) \quad} & [n] \\ {\scriptstyle j_m}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle j_n^{-1}}\big\uparrow{\scriptstyle \cong} \\ \mathrm{Hom}(I^m, I) & \xrightarrow[\mathrm{Hom}(\lambda, I)]{} & \mathrm{Hom}(I^n, I) \end{array}$$

As expressed before the statement of this definition, we must prove both that F is functorial, and that for any map λ in $\mathbb{C}$, the map $F(\lambda)$ is actually a map in $\mathcal{B}$. Functoriality is easy because F is defined by hom-ing in. Explicitly, first let $\mathrm{id}_n : I^n \to I^n$ be an identity map of $\mathbb{C}$, where we wish to show $F(\mathrm{id}_n) = \mathrm{id}_{[n]}$. We have:

$$\begin{aligned} F(\mathrm{id}_n) &= j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\mathrm{id}_n, I) \circ j_n \\ &= j_n^{-1} \circ j_n \\ &= \mathrm{id}_{[n]}, \end{aligned}$$

as desired.

Now let $\lambda : I^n \to I^m$ and $\mu : I^m \to I^p$ be maps in $\mathbb{C}$, where we show that

$$F(\mu \circ \lambda) = F(\lambda) \circ F(\mu) : [p] \to [n].$$

We have:

$$\begin{aligned} F(\mu \circ \lambda) &= j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\mu \circ \lambda, I) \circ j_p \\ &= j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\lambda, I) \circ \mathrm{Hom}_{\mathbb{C}}(\mu, I) \circ j_p \\ &= j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\lambda, I) \circ j_m \circ j_m^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\mu, I) \circ j_p \\ &= F(\lambda) \circ F(\mu), \end{aligned}$$

as desired. So $F : \mathbb{C}^{op} \to \mathcal{B}$ is indeed functorial.

Now we prove that for any map $\lambda \in \mathbb{C}$, the map $F(\lambda)$ is actually a map in $\mathcal{B}$. To do this, we first prove that F maps every basic map of $\mathbb{C}$ to a map in $\mathcal{B}$. In fact, we will prove the stronger result that F maps every basic map of $\mathbb{C}$ to the corresponding basic map of $\mathcal{B}$ (defined in Definition 2.7), i.e. that F preserves basic maps.

Proposition 3.4. *$F : \mathbb{C}^{op} \to \mathcal{B}$ preserves the basic maps of $\mathbb{C}$, in the following sense (where $n \geq 0$):*

- $F(\varepsilon_i) = e_i : [n] \to [n+1]$;
- $F(\delta_i) = d_i : [n+2] \to [n+1]$;
- $F(\sigma) = \sigma^{-1} : [n+1] \to [n+1]$;
- $F(\phi_i^\alpha) = f_i^{v(\alpha)} : [n+1] \to [n]$,

where $v : \{0,1\} \to \{-,+\}$ *is such that*

$$0 \mapsto -,$$

$$1 \mapsto +.$$

Proof. First consider any degeneracy map $\varepsilon_i : I^{n+1} \to I^n$, with $n \geq 0$. We must first show that

$$F(\varepsilon_i) : [n] \to [n+1]$$

preserves the basepoints, to be a map in $\mathcal{B}$. We show only that

$$F(\varepsilon_i)(-) = -,$$

since the case for the other basic point + is similar. We have:

$$\begin{aligned} F(\varepsilon_i)(-) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\varepsilon_i, I) \circ j_n)(-) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\varepsilon_i, I))(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_n) \\ &= j_{n+1}^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_n \circ \varepsilon_i) \\ &= j_{n+1}^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1}) \\ &= -, \end{aligned}$$

as desired, where the fourth equality follows from Proposition 2.3. Now we show that

$$F(\varepsilon_i) = e_i : [n] \to [n+1].$$

First let $1 \leq q < i$, where we must show that $F(\varepsilon_i)(q) = q$. We have:

$$\begin{aligned} F(\varepsilon_i)(q) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\varepsilon_i, I) \circ j_n)(q) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\varepsilon_i, I))(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_m) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_m \circ \varepsilon_i) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{m+1}) \\ &= q, \end{aligned}$$

as desired, where the fourth equality again follows by Proposition 2.3, since $q+1 \leq i$. Now let $i \leq q \leq n$, where we must show that $F(\varepsilon_i)(q) = q+1$. We have:

$$\begin{aligned} F(\varepsilon_i)(q) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\varepsilon_i, I) \circ j_n)(q) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\varepsilon_i, I))(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_m) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_m \circ \varepsilon_i) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_q \circ \varepsilon_{q+2} \circ \ldots \circ \varepsilon_{m+1}) \\ &= q+1, \end{aligned}$$

as desired, where the fourth equality again follows from Proposition 2.3. Thus, we have shown that $F(\varepsilon_i) = e_i$, as desired.

Now consider any diagonal map $\delta_i : I^{n+1} \to I^{n+2}$, with $n \geq 0$. First we check that

$$F(\delta_i) : [n+2] \to [n+1]$$

preserves the basepoints. We only show that

$$F(\delta_i)(-) = -,$$

since the proof for the other basic point $+$ is analogous. We calculate as follows:

$$\begin{aligned} F(\delta_i)(-) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I) \circ j_{n+2})(-) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I))(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+2}) \\ &= j_{n+1}^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+2} \circ \delta_i) \\ &= j_{n+1}^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1}) \\ &= -, \end{aligned}$$

as desired, where the fourth equality follows from Proposition 2.3. Given that $F(\delta_i)$ preserves the basepoints, we now show that

$$F(\delta_i) = d_i : [n+2] \to [n+1].$$

First let $1 \leq q \leq i$, where we must show that $F(\delta_i)(q) = q$. We have:

$$\begin{aligned} F(\delta_i)(q) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I) \circ j_{n+2})(q) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I))(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+2}) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+2} \circ \delta_i) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+1}) \\ &= q, \end{aligned}$$

as desired, where the fourth equality follows by Proposition 2.3. Now we show that $F(\delta_i)(i+1) = i$. We have:

$$\begin{aligned} F(\delta_i)(i+1) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I) \circ j_{n+2})(i+1) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I))(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_i \circ \varepsilon_{i+2} \circ \ldots \circ \varepsilon_{n+2}) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_i \circ \varepsilon_{i+2} \circ \ldots \circ \varepsilon_{n+2} \circ \delta_i) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_{i-1} \circ \varepsilon_{i+1} \circ \ldots \circ \varepsilon_{n+1}) \\ &= i, \end{aligned}$$

as desired, where the fourth equality follows by Proposition 2.3. Lastly, let $i+1 < q \leq n+2$, where we show that $F(\delta_i)(q) = q-1$. We have:

$$\begin{aligned} F(\delta_i)(q) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I) \circ j_{n+2})(q) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\delta_i, I))(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+2}) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+2} \circ \delta_i) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \varepsilon_2 \ldots \circ \varepsilon_{q-2} \circ \varepsilon_q \circ \ldots \circ \varepsilon_{n+1}) \\ &= q-1, \end{aligned}$$

as desired, where the fourth equality again follows by Proposition 2.3. Thus, we have shown that $F(\delta_i) = d_i$, as claimed.

Now consider any face map $\phi_i^\alpha : I^n \to I^{n+1}$, for $n \geq 0$. First we check that

$$F(\phi_i^\alpha) : [n+1] \to [n]$$

preserves the basepoints, by checking that $F(\phi_i^\alpha)(-) = -$, where the case for the other basepoint $+$ is analogous. We calculate as follows:

$$\begin{aligned} F(\phi_i^\alpha)(-) &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I) \circ j_{n+1})(-) \\ &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I))(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1}) \\ &= j_n^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1} \circ \phi_i^\alpha) \\ &= j_{n+1}^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1}) \\ &= -, \end{aligned}$$

as desired, where the fourth equality follows by Proposition 2.3. Now we show that

$$F(\phi_i^\alpha) = f_i^{v(\alpha)} : [n+1] \to [n],$$

where $v : \{0,1\} \to \{-,+\}$ is such that

$$0 \mapsto -,$$

$$1 \mapsto +.$$

First let $1 \leq q < i$, where we must show that $F(\phi_i^\alpha)(q) = q$. We have:

$$\begin{aligned} F(\phi_i^\alpha)(q) &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I) \circ j_{n+1})(q) \\ &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I))(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+1}) \\ &= j_n^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+1} \circ \phi_i^\alpha) \\ &= j_n^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_n) \\ &= q, \end{aligned}$$

as desired, where the fourth equality follows by Proposition 2.3. Now we show that $F(\phi_i^\alpha)(i) = v(\alpha)$. We have:

$$\begin{aligned} F(\phi_i^\alpha)(i) &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I) \circ j_{n+1})(i) \\ &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I))(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{i-1} \circ \varepsilon_{i+1} \circ \ldots \circ \varepsilon_{n+1}) \\ &= j_n^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{i-1} \circ \varepsilon_{i+1} \circ \ldots \circ \varepsilon_{n+1} \circ \phi_i^\alpha) \\ &= j_n^{-1}(\phi_0^\alpha \circ \varepsilon_1 \circ \ldots \circ \varepsilon_n) \\ &= v(\alpha), \end{aligned}$$

as desired, where the fourth equality follows by Proposition 2.3. Finally, we show that for any $i+1 \leq q \leq n+1$, $F(\phi_i^\alpha)(q) = q-1$. We have:

$$\begin{aligned} F(\phi_i^\alpha)(q) &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I) \circ j_{n+1})(q) \\ &= (j_n^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\phi_i^\alpha, I))(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+1}) \\ &= j_n^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+1} \circ \phi_i^\alpha) \\ &= j_n^{-1}(\varepsilon_1 \circ \varepsilon_2 \circ \ldots \circ \varepsilon_{q-2} \circ \varepsilon_q \circ \ldots \circ \varepsilon_n) \\ &= q-1, \end{aligned}$$

as desired, where the fourth equality follows by Proposition 2.3. Thus, we have shown that $F(\phi_i^\alpha) = f_i^{v(\alpha)}$, as claimed.

Finally, since any permutation map in $\mathbb{C}$ is a composite of mere binary swaps, it suffices to show that F preserves binary swaps. So consider any binary swap $\sigma : I^{n+1} \to I^{n+1}$ for $n \geq 0$ (induced by $\sigma : \{1, \ldots, n+1\} \to \{1, \ldots, n+1\}$). First we check that

$$F(\sigma) : [n+1] \to [n+1]$$

preserves the basepoints: we only show that $F(\sigma)(-) = -$, since the proof for the other basic point $+$ is analogous. We have:

$$\begin{aligned} F(\sigma)(-) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I) \circ j_{n+1})(-) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I))(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1}) \\ &= j_{n+1}^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1} \circ \sigma) \\ &= j_{n+1}^{-1}(\phi_0^0 \circ \varepsilon_1 \circ \ldots \circ \varepsilon_{n+1}) \\ &= -, \end{aligned}$$

as desired, where the fourth equality follows from Proposition 2.3. Now we show that

$$F(\sigma) = \sigma^{-1} : [n+1] \to [n+1].$$

We have two cases to consider, depending on whether n is swapped or not. We only consider the case where n is swapped, because the other case is proved similarly. So suppose that n is swapped and $\sigma(n) = n+1$. First we show that $F(\sigma)(n+1) = n$; so we have

$$\begin{aligned} F(\sigma)(n+1) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I) \circ j_{n+1})(n+1) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I))(\varepsilon_1 \circ \ldots \circ \varepsilon_n) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \ldots \circ \varepsilon_n \circ \sigma) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \ldots \circ \varepsilon_{n-1} \circ \varepsilon_{n+1}) \\ &= n, \end{aligned}$$

as desired, since $\varepsilon_n \circ \sigma = \varepsilon_{n+1}$ (see Proposition 2.4). Now we show that $F(\sigma)(n) = n+1$. We have:

$$\begin{aligned} F(\sigma)(n) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I) \circ j_{n+1})(n) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I))(\varepsilon_1 \circ \ldots \circ \varepsilon_{n-1} \circ \varepsilon_{n+1}) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \ldots \circ \varepsilon_{n-1} \circ \varepsilon_{n+1} \circ \sigma) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \ldots \circ \varepsilon_{n-1} \circ \varepsilon_n) \\ &= n+1, \end{aligned}$$

as desired, since $\varepsilon_{n+1} \circ \sigma = \varepsilon_n$ by Proposition 2.4. Now let $1 \le q < n$, where we show that $F(\sigma)(q) = q$. We have:

$$\begin{aligned} F(\sigma)(q) &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I) \circ j_{n+1})(q) \\ &= (j_{n+1}^{-1} \circ \mathrm{Hom}_{\mathbb{C}}(\sigma, I))(\varepsilon_1 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+1}) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_{n+1} \circ \sigma) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_n \circ \varepsilon_n) \\ &= j_{n+1}^{-1}(\varepsilon_1 \circ \ldots \circ \varepsilon_{q-1} \circ \varepsilon_{q+1} \circ \ldots \circ \varepsilon_n \circ \varepsilon_{n+1}) \\ &= q, \end{aligned}$$

as desired, where the fifth equality follows from Proposition 2.3, and because $\varepsilon_{n+1} \circ \sigma = \varepsilon_n$ (see Proposition 2.4). So in the case where $\sigma(n) = n+1$, we have shown that $F(\sigma) = \sigma$, as desired.

This completes the proof of Proposition 3.4, that $F : \mathbb{C}^{op} \to \mathcal{B}$ preserves the basic maps of $\mathbb{C}$.

□

Now we can actually prove that the functor $F : \mathbb{C}^{op} \to \mathcal{B}$ lands in the category $\mathcal{B}$.

Proposition 3.5. *For any map $\lambda : I^n \to I^m$ in $\mathbb{C}$ (for any $n, m \ge 0$), the map $F(\lambda) : [m] \to [n]$ is a map in $\mathcal{B}$.*

Proof. Let $\lambda : I^n \to I^m$ be a map in $\mathbb{C}$, where we show that $F(\lambda) : [m] \to [n]$ is in fact a map in $\mathcal{B}$. But now this easily follows because (by Proposition 2.5) λ

can be factorized as a composite of basic maps in $\mathbb{C}$, and we have just shown in Proposition 3.4 that F maps each such basic map in $\mathbb{C}$ to a (basic) map in $\mathcal{B}$, so that by functoriality of F (which was proven before the statement of Proposition 3.4), it follows that $F(\lambda)$ is a composite of maps in $\mathcal{B}$ and hence is a map in $\mathcal{B}$. So the functor $F : \mathbb{C}^{op} \to \mathcal{B}$ lands in the category $\mathcal{B}$, as was to be shown. □

Just as with the functor F, the difficulty in defining $G : \mathcal{B}^{op} \to \mathbb{C}$ lies in defining G on the maps of $\mathcal{B}$ so that G actually lands in $\mathbb{C}$. To do this, we take advantage of the first isomorphism

$$i_m : I^m \to \mathrm{Hom}_{\mathcal{B}}([m],[0])$$

defined in Proposition 3.1, as well as hom-ing into the fixed object $[0]$. We present the definition of the functor G as follows:

Definition 3.6. We define the functor $G : \mathcal{B}^{op} \to \mathbb{C}$ as follows:

- *Objects*: For any object $[n] \in \mathcal{B}$, we set

$$G([n]) = I^n.$$

- *Maps*: For any map $g : [n] \to [m]$ in $\mathcal{B}$, we set

$$G(g) = i_n^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(g,[0]) \circ i_m$$

$$: I^m \to \mathrm{Hom}_{\mathcal{B}}([m],[0]) \to \mathrm{Hom}_{\mathcal{B}}([n],[0]) \to I^n,$$

as in the following commutative diagram:

$$\begin{array}{ccc} I^m & \xrightarrow{\quad G(g) \quad} & I^n \\ {\scriptstyle i_m}\downarrow\cong & & {\scriptstyle i_n^{-1}}\uparrow\cong \\ \mathrm{Hom}([m],[0]) & \xrightarrow[\mathrm{Hom}(g,[0])]{} & \mathrm{Hom}([n],[0]) \end{array}$$

The proof that G is actually functorial is exactly parallel to the proof that $F : \mathbb{C}^{op} \to \mathcal{B}$ is functorial, since G is defined on the maps of $\mathcal{B}$ by hom-ing into the fixed object $[0]$. To show that G lands in $\mathbb{C}$, i.e. that $G(g)$ is a actually a map in $\mathbb{C}$ for every map $g \in \mathcal{B}$, we first show that G maps every basic map of $\mathcal{B}$ to the corresponding basic map of $\mathbb{C}$, i.e. that G preserves basic maps.

Proposition 3.7. *$G : \mathcal{B}^{op} \to \mathbb{C}$ preserves the basic maps of $\mathcal{B}$, in the following sense (for $n \geq 0$):*

- $G(e_i) = \varepsilon_i : I^{n+1} \to I^n$.
- $G(d_i) = \delta_i : I^{n+1} \to I^{n+2}$.
- $G(s) = s^{-1} : I^{n+1} \to I^{n+1}$.

- *$G(f_i^\alpha) = \phi_i^{w(\alpha)} : I^n \to I^{n+1}$, where $w : \{-,+\} \to \{0,1\}$ is such that*

$$- \mapsto 0,$$

$$+ \mapsto 1.$$

Proof. First consider any co-degeneracy map $\varepsilon_i : [n] \to [n+1]$, where we show that

$$G(e_i) = \varepsilon_i : I^{n+1} \to I^n.$$

So let $(x_1, \dots, x_{n+1}) \in I^{n+1}$, where we show that

$$G(e_i)(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}).$$

Well, we have that

$$\begin{aligned} G(e_i)(x_1, \dots, x_{n+1}) &= (i_n^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(e_i, [0]) \circ i_{n+1})(x_1, \dots, x_{n+1}) \\ &= i_n^{-1}(i_{n+1}(x_1, \dots, x_{n+1}) \circ e_i) \\ &= (w((i_{n+1}(x_1, \dots, x_{n+1}) \circ e_i)(1)), \dots, w((i_{n+1}(x_1, \dots, x_{n+1}) \circ e_i)(n))). \end{aligned}$$

Now we wish to show that for any $1 \leq m < i$,

$$w((i_{n+1}(x_1, \dots, x_{n+1}) \circ e_i)(m)) = x_m,$$

and for any $1 \leq m \leq n$,

$$w((i_{n+1}(x_1, \dots, x_{n+1}) \circ e_i)(m)) = x_{m+1}.$$

First consider $1 \leq m < i$: then we have

$$\begin{aligned} w((i_{n+1}(x_1, \dots, x_{n+1}) \circ e_i)(m)) &= w((i_{n+1}(x_1, \dots, x_{n+1})(m)) \\ &= w(w^{-1}(x_m)) \\ &= x_m, \end{aligned}$$

as desired. Now consider $i \leq m \leq n$: then we have

$$\begin{aligned} w((i_{n+1}(x_1, \dots, x_{n+1}) \circ e_i)(m)) &= w((i_{n+1}(x_1, \dots, x_{n+1})(m+1)) \\ &= w(w^{-1}(x_{m+1})) \\ &= x_{m+1}, \end{aligned}$$

as desired. Thus, we have shown that

$$G(e_i)(x_1, \dots, x_{n+1}) = (x_1, \dots, \hat{x}_i, \dots, x_{n+1}),$$

so that $G(e_i) = \varepsilon_i$, as claimed.

Now consider any co-diagonal map $d_i : [n+2] \to [n+1]$, where we show that

$$G(d_i) = \delta_i : I^{n+1} \to I^{n+2}.$$

So let $(x_1, \dots, x_{n+1}) \in I^{n+1}$, where we show that

$$G(d_i)(x_1, \dots, x_{n+1}) = (x_1, \dots, x_i, x_i, \dots, x_{n+1}) \in I^{n+2}.$$

Well, we have

$$\begin{aligned} G(d_i)(x_1, \dots, x_{n+1}) &= (i_{n+2}^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(d_i, [0]) \circ i_{n+1})(x_1, \dots, x_{n+1}) \\ &= i_{n+2}^{-1}(i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i) \\ &= (w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(1)), \dots, w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(n+2)). \end{aligned}$$

Now we wish to show that for any $1 \le m \le i$, we have

$$w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(m)) = x_m,$$

that we have

$$w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(i+1)) = x_i,$$

and that for any $i + 1 < m \le n + 2$, we have

$$w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(m)) = x_{m-1}.$$

First consider $1 \le m \le i$: then we have

$$\begin{aligned} w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(m)) &= w((i_{n+1}(x_1, \dots, x_{n+1})(m)) \\ &= w(w^{-1}(x_m)) \\ &= x_m, \end{aligned}$$

as desired. Now consider $i + 1$: then we have

$$\begin{aligned} w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(i+1)) &= w((i_{n+1}(x_1, \dots, x_{n+1})(i)) \\ &= w(w^{-1}(x_i)) \\ &= x_i, \end{aligned}$$

as desired. Lastly, consider $i + 1 < m \le n + 2$: then we have

$$\begin{aligned} w((i_{n+1}(x_1, \dots, x_{n+1}) \circ d_i)(m)) &= w((i_{n+1}(x_1, \dots, x_{n+1})(m-1)) \\ &= w(w^{-1}(x_{m-1})) \\ &= x_{m-1}, \end{aligned}$$

again as desired. Thus, we have shown that

$$G(d_i)(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_i, x_i, \ldots, x_{n+1}),$$

whereby $G(d_i) = \delta_i$, as claimed.

Now consider any (co-)permutation map $s : [n+1] \to [n+1]$, where we show that

$$G(s) = s^{-1} : I^{n+1} \to I^{n+1}.$$

So let $(x_1, \ldots, x_{n+1}) \in I^{n+1}$, where we show that

$$G(s)(x_1, \ldots, x_{n+1}) = (x_{s(1)}, \ldots, x_{s(n+1)}).$$

Well, we have:

$$\begin{aligned} G(s)(x_1, \ldots, x_{n+1}) &= (i_{n+1}^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(s, [0]) \circ i_{n+1})(x_1, \ldots, x_{n+1}) \\ &= i_{n+1}^{-1}(i_{n+1}(x_1, \ldots, x_{n+1}) \circ s) \\ &= (w((i_{n+1}(x_1, \ldots, x_{n+1}) \circ s)(1)), \ldots, w((i_{n+1}(x_1, \ldots, x_{n+1}) \circ s)(n+1))) \end{aligned}$$

Now we wish to show that for any $1 \leq m \leq n+1$, we have

$$w((i_{n+1}(x_1, \ldots, x_{n+1}) \circ s)(m)) = x_{s(m)}.$$

Well, we have

$$\begin{aligned} w((i_{n+1}(x_1, \ldots, x_{n+1}) \circ s)(m)) &= w((i_{n+1}(x_1, \ldots, x_{n+1})(s(m))) \\ &= w(w^{-1}(x_{s(m)})) \\ &= x_{s(m)}, \end{aligned}$$

as desired. Thus, we have shown that

$$G(s)(x_1, \ldots, x_{n+1}) = (x_{s(1)}, \ldots, x_{s(n+1)}),$$

so that we have $G(s) = s^{-1}$, as claimed.

Lastly, consider any co-face map $f_i^\alpha : [n+1] \to [n]$ (with $\alpha \in \{-, +\}$), where we show that

$$G(f_i^\alpha) = \phi_i^{w(\alpha)} : I^n \to I^{n+1}.$$

So let $(x_1, \ldots, x_n) \in I^n$, where we show that

$$G(f_i^\alpha)(x_1, \ldots, x_n) = (x_1, \ldots, w(\alpha), x_i, \ldots, x_n) \in I^{n+1}.$$

Well, we have that

$$\begin{aligned} G(f_i^\alpha)(x_1,\ldots,x_n) &= (i_{n+1}^{-1} \circ \mathrm{Hom}_{\mathcal{B}}(f_i^\alpha,[0]) \circ i_n)(x_1,\ldots,x_n) \\ &= i_{n+1}^{-1}(i_n(x_1,\ldots,x_n) \circ f_i^\alpha) \\ &= (w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(1)),\ldots,w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(n+1))). \end{aligned}$$

Now we wish to show that for any $1 \leq m \leq i-1$, we have

$$w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(m)) = x_m,$$

that

$$w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(i)) = w(\alpha),$$

and that for any $i+1 \leq m \leq n+1$, we have

$$w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(m)) = x_{m-1}.$$

First consider $1 \leq m \leq i-1$: then we have

$$\begin{aligned} w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(m)) &= w((i_n(x_1,\ldots,x_n)(m)) \\ &= w(w^{-1}(x_m)) \\ &= x_m, \end{aligned}$$

as desired. Now consider i: then we have

$$\begin{aligned} w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(i)) &= w((i_n(x_1,\ldots,x_n)(\alpha)) \\ &= w(\alpha) \end{aligned}$$

(since $i_n(x_1,\ldots,x_n) : [n] \to [0]$ preserves basepoints), as desired. Lastly, consider $i+1 \leq m \leq n+1$: then we have

$$\begin{aligned} w((i_n(x_1,\ldots,x_n)\circ f_i^\alpha)(m)) &= w((i_n(x_1,\ldots,x_n)(m-1)) \\ &= w(w^{-1}(x_{m-1})) \\ &= x_{m-1}, \end{aligned}$$

as desired. Thus, we have shown that

$$G(f_i^\alpha)(x_1,\ldots,x_n) = (x_1,\ldots,w(\alpha),x_i,\ldots,x_n) \in I^{n+1},$$

so that $G(f_i^\alpha) = \phi_i^{w(\alpha)}$, as claimed.

This completes the proof that $G : \mathcal{B}^{op} \to \mathbb{C}$ preserves the basic maps of $\mathbb{C}$.

□

Now it easily follows that G lands in $\mathbb{C}$, because if $g : [n] \to [m]$ is any map in $\mathcal{B}$, then we know that $G(g) : I^m \to I^n$ is a map in $\mathbb{C}$, as follows.

Proposition 3.8. *For any map $g : [n] \to [m]$ in $\mathcal{B}$ (for any $n, m \geq 0$), the map $G(g) : I^m \to I^n$ is a map in $\mathbb{C}$.*

Proof. We know that g factorizes as $g = e \circ d \circ s \circ f$ in $\mathcal{B}$ (see Proposition 2.8), where e is a composite of co-degeneracy maps, d is a composite of co-diagonal maps, s is a (co-)permutation map, and f is a composite of co-face maps, and we have just shown that the image of any such map under G is a (basic) map in $\mathbb{C}$ (and that G is functorial). So G does in fact land in $\mathbb{C}$. □

So we have now finished constructing the two contravariant functors $F : \mathbb{C} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathbb{C}$, and now it remains to demonstrate that these two functors are actually mutually inverse, thereby establishing that $\mathbb{C}$ is dual to $\mathcal{B}$. This is the content of the following theorem.

Theorem 3.9. *The functors $F : \mathbb{C}^{op} \to \mathcal{B}$ and $G : \mathcal{B}^{op} \to \mathbb{C}$ are mutually inverse, thereby establishing the desired duality-isomorphism:*

$$\mathbb{C} \cong \mathcal{B}^{op}.$$

Proof. That the object parts of F and G are mutually inverse is trivial from the definitions of F and G:

$$GF(I^n) = G([n]) = I^n,$$
$$FG([n]) = F(I^n) = [n].$$

Now we show that the morphism parts of F and G are mutually inverse.

First let $\lambda : I^n \to I^m$ be any map in $\mathbb{C}$, where we show that $G(F(\lambda)) = \lambda$. First, let

$$\lambda = \phi_{q_j}^{\alpha_j} \circ \ldots \circ \phi_{q_1}^{\alpha_1} \circ \sigma \circ \delta_{i_k} \circ \ldots \circ \delta_{i_1} \circ \varepsilon_{r_s} \circ \ldots \circ \varepsilon_{r_1}$$

be a factorization of $\lambda \in \mathbb{C}$ (see Proposition 2.5). Then we calculate as follows:

$$\begin{aligned} GF(\lambda) &= GF(\phi_{q_j}^{\alpha_j} \circ \ldots \circ \phi_{q_1}^{\alpha_1} \circ \sigma \circ \delta_{i_k} \circ \ldots \circ \delta_{i_1} \circ \varepsilon_{r_s} \circ \ldots \circ \varepsilon_{r_1}) \\ &= GF(\phi_{q_j}^{\alpha_j}) \circ \ldots \circ GF(\phi_{q_1}^{\alpha_1}) \circ GF(\sigma) \circ GF(\delta_{i_k}) \circ \ldots \circ GF(\delta_{i_1}) \circ GF(\varepsilon_{r_s}) \circ \ldots \circ GF(\varepsilon_{r_1}) \\ &= G(f_{q_j}^{v(\alpha_j)}) \circ \ldots \circ G(f_{q_1}^{v(\alpha_1)}) \circ G(\sigma^{-1}) \circ G(d_{i_k}) \circ \ldots \circ G(d_{i_1}) \circ \ldots \circ G(e_{r_s}) \circ \ldots \circ G(e_{r_1}) \\ &= \phi_{q_j}^{\alpha_j} \circ \ldots \circ \phi_{q_1}^{\alpha_1} \circ \sigma \circ \delta_{i_k} \circ \ldots \circ \delta_{i_1} \circ \varepsilon_{r_s} \circ \ldots \circ \varepsilon_{r_1} \\ &= \lambda, \end{aligned}$$

as desired, where the second equality follows from functoriality of F and G and covariance of GF, the third equality follows because F preserves basic maps, and the fourth equality follows because G preserves basic maps.

If $g : [n] \to [m]$ is any map in $\mathcal{B}$, then exactly parallel reasoning shows that $FG(g) = g$. So we have shown that $FG = \mathrm{id}_{\mathcal{B}}$ and $GF = \mathrm{id}_{\mathbb{C}}$, whereby we have the duality-isomorphism

$$\mathbb{C} \cong \mathcal{B}^{op},$$

as claimed.

□

One important consequence of this duality $\mathbb{C} \cong \mathcal{B}^{op}$ is the following.

Corollary 3.10. *The Cartesian category $\mathbb{C}$ of cubes has finite products.*

Proof. First, $I^0 = \{*\}$ is the terminal object and for any $n, m \geq 0$ we have $I^n \times I^m = I^{n+m}$. To see this, we note that $\mathcal{B}$ has finite coproducts: the initial object is clearly $[0] = \{-, +\}$, while for any $n, m \geq 0$, it is easily seen that $[n] + [m] = [n+m]$. But then $G([0]) = I^0$ is the terminal object of $\mathbb{C}$, while

$$I^n \times I^m = G([n]) \times G([m]) = G([n] + [m]) = G([n+m]) = I^{n+m}.$$

□

This concludes Section 3.1, where we have proven that the Cartesian category of cubes $\mathbb{C}$ is dual to the category $\mathcal{B}$ of finite strictly bipointed sets.

3.3 Lawvere Duality

Recall that an algebraic theory $\mathbb{T}$ is given by a signature consisting of term-forming operations (constants and function symbols), as well as a set of equations between terms as axioms (where the terms are formed from the signature). Given such an algebraic theory $\mathbb{T}$, we can then form the *syntactic category* $\mathcal{C}_{\mathbb{T}}$ of $\mathbb{T}$, defined as follows:

Definition 3.11. The syntactic category $\mathcal{C}_{\mathbb{T}}$ of the algebraic theory $\mathbb{T}$ is defined as follows:

- *Objects*: The objects are contexts, i.e. finite sequences of variables (for $n \geq 0$)

$$[x_1, \ldots, x_n].$$

- *Maps*: A map from $[x_1, \ldots, x_m]$ to $[x_1, \ldots, x_n]$ is an n-tuple of terms (of the signature) $(t_1, \ldots, t_n)$, where for any $1 \leq k \leq n$, we have

$$x_1, \ldots, x_m \mid t_k,$$

  i.e. all of the free variables of the term t_k come from the context $x_1, \ldots, x_m$.

  Two morphisms

$$(t_1, \ldots, t_n), (s_1, \ldots, s_n) : [x_1, \ldots, x_m] \to [x_1, \ldots, x_n]$$

  are equal iff for every $1 \leq k \leq n$, we have

$$\mathbb{T} \vdash t_k = s_k.$$

Strictly speaking, morphisms are equivalence classes of terms in context.

Composition of morphisms is defined as follows: given two composable morphisms

$$(t_1, \dots, t_m) : [x_1, \dots, x_k] \to [x_1, \dots, x_m],$$
$$(s_1, \dots, s_n) : [x_1, \dots, x_m] \to [x_1, \dots, x_n],$$

we let the composite morphism

$$(s_1, \dots, s_n) \circ (t_1, \dots, t_m) : [x_1, \dots, x_k] \to [x_1, \dots, x_n]$$

be the morphism $(r_1, \dots, r_n)$ such that for any $1 \leq i \leq n$, we have

$$r_i \doteq s_i[t_1, \dots, t_m / x_1, \dots, x_m],$$

i.e. the i^{th} component of $(r_1, \dots, r_n)$ is the term obtained by simultaneously substituting in s_i the terms $t_1, \dots, t_m$ for the variables $x_1, \dots, x_m$.

The identity morphism on $[x_1, \dots, x_n]$ is the n-tuple $(x_1, \dots, x_n)$.

Now, given an algebraic theory $\mathbb{T}$, one can also form the category $\mathrm{Mod}(\mathbb{T})$ of all set-theoretic models of $\mathbb{T}$, i.e. all models of $\mathbb{T}$ in the category Sets. One can then restrict this category $\mathrm{Mod}(\mathbb{T})$ to the full subcategory $\mathrm{Mod}_{\mathrm{fgf}}(\mathbb{T})$ of all finitely generated, free models in the category Sets. Then, one of the central results of Lawvere duality theory states that

$$\mathcal{C}_{\mathbb{T}} \simeq \mathrm{Mod}_{\mathrm{fgf}}(\mathbb{T})^{op},$$

i.e. that the syntactic category of $\mathbb{T}$ is dual to the category of finitely generated, free models of $\mathbb{T}$ in Sets ([9]).

Consider now the simple algebraic theory $\mathbb{T}_b$ with just two constants and no equations (b for *bipointed*). As we will easily show, the category of finitely generated, free models of $\mathbb{T}_b$ in Sets is *exactly* the category $\mathcal{B}$ of finite strictly bipointed sets:

$$\mathrm{Mod}_{\mathrm{fgf}}(\mathbb{T}_b) = \mathcal{B}.$$

So by the aforementioned result of Lawvere duality theory, we find that

$$\mathcal{C}_{\mathbb{T}_b} \simeq \mathcal{B}^{op}.$$

Then since we now know that $\mathcal{B}^{op} \cong \mathbb{C}$, we obtain that

$$\mathcal{C}_{\mathbb{T}_b} \simeq \mathbb{C},$$

i.e. that the syntactic category of the theory $\mathbb{T}_b$ with just two constants and no equations is equivalent to the Cartesian cube category $\mathbb{C}$. We can describe the correspondence between $\mathcal{C}_{\mathbb{T}}$ and $\mathbb{C}$ as follows, where

$$H : \mathbb{C} \to \mathcal{C}_{\mathbb{T}_b}$$

is a functor witnessing the equivalence:

1. *Objects*: For any $n \geq 0$, we have

$$H(I^n) = [x_1, \ldots, x_n].$$

2. *Maps*: We describe the correspondence for the basic maps of $\mathbb{C}$ as follows:

   - For any degeneracy map $\varepsilon_i : I^{n+1} \to I^n$ (for $n \geq 0$), we have

   $$H(\varepsilon_i) = (x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}) : [x_1, \ldots, x_{n+1}] \to [x_1, \ldots, x_n].$$

   - For any face map $\phi_i^\alpha : I^n \to I^{n+1}$ (for $n \geq 0, \alpha \in \{0, 1\}$), we have

   $$H(\phi_i^\alpha) = (x_1, \ldots, \alpha', \ldots, x_{n+1}) : [x_1, \ldots, x_n] \to [x_1, \ldots, x_{n+1}],$$

   where α' is one of the two constants of $\mathbb{T}_b$, depending on whether α is 0 or 1.

   - For any diagonal map $\delta_i : I^n \to I^{n+1}$ (for $n \geq 1$), we have

   $$H(\delta_i) = (x_1, \ldots, x_i, x_i, \ldots, x_n) : [x_1, \ldots, x_n] \to [x_1, \ldots, x_{n+1}].$$

   - For any permutation map $\sigma : I^n \to I^n$ (for $n \geq 1$), we have

   $$H(\sigma) = (x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}) : [x_1, \ldots, x_n] \to [x_1, \ldots, x_n].$$

Lastly, we confirm the (easy) result that the category $\mathrm{Mod}_{\mathrm{fgf}}(\mathbb{T}_b)$ of finite generated, free models of $\mathbb{T}_b$ in Sets is exactly the category $\mathcal{B}$ of finite strictly bipointed sets.

Proposition 3.12. $\mathrm{Mod}_{\mathrm{fgf}}(\mathbb{T}_b) = \mathcal{B}$.

Proof. First let $\mathrm{Mod}(\mathbb{T}_b)$ be the category of *all* models of $\mathbb{T}_b$ in Sets. So an object of $\mathrm{Mod}(\mathbb{T}_b)$ is just a triple (X, a, b) with X a (possibly infinite) set and basepoints $a, b \in X$ (not necessarily distinct), while a map $f : (X, a, b) \to (Y, c, d)$ is a function $f : X \to Y$ between the underlying sets that preserves the basepoints (i.e. $f(a) = c$ and $f(b) = d$). Then there is a forgetful functor

$$U : \mathrm{Mod}(\mathbb{T}_b) \to \mathrm{Sets}$$

with

$$U(X, a, b) = X,$$
$$U(f) = f : X \to Y.$$

Now we show that U has a left adjoint

$$F : \mathrm{Sets} \to \mathrm{Mod}(\mathbb{T}_b),$$

which will map any set to the free model of $\mathbb{T}_b$ generated by that set. To show that such a left adjoint F exists, it suffices to provide for each set S an object

$F(S) \in \text{Mod}(\mathbb{T}_b)$ and a universal arrow $\eta_S : S \to U(F(S))$, universal from S to the functor U. So let S be any set, where we set

$$F(S) = (S \cup \{a_S, b_S\}, a_S, b_S),$$

where $a_S, b_S \notin S$ and $a_S \neq b_S$. So $F(S)$ is a strictly bipointed set. Now let $\eta_S : S \to U(F(S))$ be that function

$$\eta_S : S \to S \cup \{a_S, b_S\}$$

defined by

$$\eta_S(s) = s$$

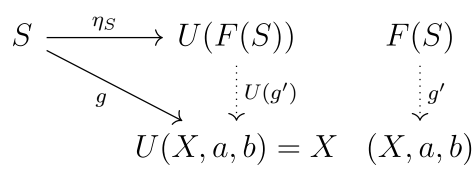
for any $s \in S$. To show that η_S is universal from S to U, let $g : S \to U(X, a, b) = X$, where we show that there is a unique basepoint-preserving function

$$g' : F(S) = (S \cup \{a_S, b_S\}, a_S, b_S) \to (X, a, b)$$

such that

$$U(g') \circ \eta_S = g' \circ \eta_S = g,$$

as in the following diagram:



Well, we define g' to be that basepoint-preserving function such that $g'(s) = g(s)$ for any $s \in S$. Then clearly we have $g' \circ \eta_S = g : S \to X$, and it is easy to see that g' is unique with this property. Therefore, we obtain a functor

$$F : \text{Sets} \to \text{Mod}(\mathbb{T}_b)$$

that is left adjoint to $U : \text{Mod}(\mathbb{T}_b) \to \text{Sets}$, $F \dashv U$, where the action of F on arrows is defined as follows: for any function $f : S \to S'$, we have $F(f) : F(S) \to F(S')$ as that basepoint-preserving function such that for any $s \in S$, $F(f(s)) = f(s) \in S'$.

We can now restrict $F : \text{Sets} \to \text{Mod}(\mathbb{T}_b)$ to the full subcategory Ord_{fin} of Sets whose objects are all the finite ordinal numbers. Then for any finite ordinal number n, we have

$$F(n) \cong [n] = (\{1, \ldots, n, -, +\}, -, +),$$

so that the image of Ord_{fin} under F is (isomorphic to) the category $\mathcal{B}$ of finite strictly bipointed sets. Then since the image of Ord_{fin} under F is just the category $\text{Mod}_{\text{fgf}}(\mathbb{T}_b)$ of finitely generated, free models of $\mathbb{T}_b$ (because a finitely-generated, free model of $\mathbb{T}_b$ is necessarily finite, since there are no function symbols in the signature of $\mathbb{T}_b$), it follows that

$$\text{Mod}_{\text{fgf}}(\mathbb{T}_b) = \mathcal{B},$$

as desired.

□

This concludes Chapter 3, in which we have proven the duality of the Cartesian category of cubes $\mathbb{C}$ and the category $\mathcal{B}$ of finite strictly bipointed sets by means of a new instance of Stone-type duality given by hom-ing into the two-element set as fixed object. We also used one of the consequences of Lawvere duality theory to prove that the Cartesian category of cubes $\mathbb{C}$ is in fact equivalent to the syntactic category $\mathbb{C}_{\mathbb{T}_b}$ of the algebraic theory $\mathbb{T}_b$ with just two constants and no equations.

Chapter 4

Classifying Properties of Cubes and Bipointed Sets

In this chapter we prove some classifying properties of the Cartesian cube category $\mathbb{C}$ and the category $\mathcal{B}$ of finite strictly bipointed sets. Specifically, we prove that $\mathbb{C}$ is the *free finite-product category on an interval*, by showing that $\mathcal{B}$ has the dual property of being the free finite-*coproduct* category on a *co-interval* (the notions of interval and co-interval will be defined shortly). We then introduce an expanded version of the category $\mathcal{B}$, namely the category $\mathcal{B}_w$ of finite *weakly* bipointed sets, whose objects are finite sets equipped with two (possibly equal) basepoints. We then show that this category $\mathcal{B}_w$ has the classifying property of being the free finite-*colimit* category on a co-interval, so that $\mathbb{C}_w \doteq \mathcal{B}_w^{op}$ is the free finite-limit category on an interval.

4.1 Classifying Properties of $\mathbb{C}$ and $\mathcal{B}$

In the first section of this chapter we will show that the category $\mathcal{B}$ of finite strictly bipointed sets has the classifying property of being the free finite-coproduct category on a co-interval, which we will define shortly. Given the duality between $\mathcal{B}$ and $\mathbb{C}$ proved in the preceding chapter, we will then infer that the Cartesian cube category $\mathbb{C}$ has the dual classifying property of being the free finite-product category on an interval. First, we define an *interval* (X, a, b) in a category $\mathcal{C}$ with terminal object 1 to be an object $X \in \mathcal{C}$ together with two points $a, b : 1 \to X$ in $\mathcal{C}$. Before we present the definition of being the free finite-product category on an interval, we first make a few more preliminary definitions. Let $\mathcal{C}$ be any category with finite products and a distinguished interval $(I, \top, \bot)$, so that $\top, \bot : 1 \to I$. Also, for any category $\mathcal{D}$ with finite products, we define the category $\text{Int}(\mathcal{D})$ of intervals of D in the obvious way, as follows:

- *Objects*: Intervals (X, a, b) of $\mathcal{D}$, where $a, b : 1 \to X$.
- *Maps*: A map $f : (X, a, b) \to (Y, c, d)$ between intervals of $\mathcal{D}$ is a map $f : X \to$

Y of $\mathcal{D}$ such that

$$f \circ a = c : 1 \to Y,$$
$$f \circ b = d : 1 \to Y.$$

Also, for any category $\mathcal{D}$ with finite products, we have the category $\mathrm{FP}(\mathcal{C}, \mathcal{D})$ of finite-product-preserving functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations. Finally, we have the canonical evaluation functor

$$\mathrm{eval} : \mathrm{FP}(\mathcal{C}, \mathcal{D}) \to \mathrm{Int}(\mathcal{D})$$

defined at a finite-product-preserving functor $F : \mathcal{C} \to \mathcal{D}$ by

$$\mathrm{eval}(F) = (F(I), F(\top), F(\bot))$$

as an interval in $\mathcal{D}$, with

$$F(\top), F(\bot) : 1 \cong F(1) \to F(X),$$

and defined at a natural transformation $\theta : F \to G$ of finite-product-preserving functors F, G by

$$\mathrm{eval}(\theta) = \theta_I : (F(I), F(\top), F(\bot)) \to (G(I), G(\top), G(\bot)),$$

where the required interval homomorphism commutativity conditions $\theta_I \circ F(\top) = G(\top)$, $\theta_I \circ F(\bot) = G(\bot)$ follow by naturality of θ. In summary, $\mathrm{eval} : \mathrm{FP}(\mathcal{C}, \mathcal{D}) \to \mathrm{Int}(\mathcal{D})$ evaluates a finite-product-preserving functor at the distinguished interval $(I, \top, \bot)$ of $\mathcal{C}$ to produce an interval in $\mathcal{D}$. Strictly speaking, the functor eval should be indexed by the categories $\mathcal{C}$, $\mathcal{D}$, and the distinguished interval of $\mathcal{C}$, but we suppress these indices where no confusion should arise. Now we can finally define what it means for a category $\mathcal{C}$ to be the free finite-product category on an interval:

Definition 4.1. A category $\mathcal{C}$ is the *free finite-product category on an interval* if $\mathcal{C}$ has finite products and a distinguished interval $(I, \top, \bot)$ so that for any category $\mathcal{D}$ with finite products, the canonical evaluation functor

$$\mathrm{eval} : \mathrm{FP}(\mathcal{C}, \mathcal{D}) \to \mathrm{Int}(\mathcal{D})$$

of evaluation at the distinguished interval $(I, \top, \bot)$ of $\mathcal{C}$ is an equivalence of categories.

This equivalence of categories will be natural in $\mathcal{D}$, in the sense that for any finite-product-preserving functor $H : \mathcal{D} \to \mathcal{D}'$ (where $\mathcal{D}'$ is a category with finite products), the following diagram commutes:

$$\begin{array}{ccc} \mathrm{FP}(\mathcal{C}, \mathcal{D}) & \xrightarrow{\mathrm{eval}} & \mathrm{Int}(\mathcal{D}) \\ {\scriptstyle \mathrm{FP}(\mathcal{C}, H)} \downarrow & & \downarrow {\scriptstyle \mathrm{Int}(H)} \\ \mathrm{FP}(\mathcal{C}, \mathcal{D}') & \xrightarrow[\mathrm{eval}]{} & \mathrm{Int}(\mathcal{D}'), \end{array}$$

where

$$\mathrm{FP}(\mathcal{C}, H) : \mathrm{FP}(\mathcal{C}, \mathcal{D}) \to \mathrm{FP}(\mathcal{C}, \mathcal{D}')$$

is defined by post-composition with $H : \mathcal{D} \to \mathcal{D}'$, while the induced functor

$$\mathrm{Int}(H) : \mathrm{Int}(\mathcal{D}) \to \mathrm{Int}(\mathcal{D}')$$

is defined in the obvious way by

$$(X, a, b) \mapsto (H(X), H(a), H(b)),$$

$$[f : (X, a, b) \to (Y, c, d)] \mapsto [H(f) : (H(X), H(a), H(b)) \to (H(Y), H(c), H(d))]\,,$$

where $H(a), H(b) : 1 \cong H(1) \to H(X)$ (since H preserves finite products). It is obvious that this naturality diagram commutes (for any finite-product-preserving $H : \mathcal{D} \to \mathcal{D}'$) because for any finite-product-preserving functor $F : \mathcal{C} \to \mathcal{D}$, going either way around the diagram yields the $\mathcal{D}'$ interval $(HF(I), HF(\top), HF(\bot))$.

Spelling out what Definition 4.1 means, suppose that $\mathcal{C}$ is the free finite-product category on an interval (with distinguished interval $(I, \top, \bot)$). Then for any category $\mathcal{D}$ with finite products, the canonical evaluation functor

$$\mathrm{eval} : \mathrm{FP}(\mathcal{C}, \mathcal{D}) \to \mathrm{Int}(\mathcal{D})$$

creates an equivalence of categories, i.e. is full, faithful, and essentially surjective. Essential surjectivity means that if we have any interval (X, a, b) in $\mathcal{D}$ (so that $a, b : 1 \to X$), then there exists a finite-product-preserving functor $F : \mathcal{C} \to \mathcal{D}$ such that

$$\mathrm{eval}(F) = (F(I), F(\top), F(\bot)) \cong (X, a, b),$$

which means that there is an isomorphism $F^* : F(I) \to X$ in $\mathcal{D}$ that makes the following diagrams commute:

$$\begin{array}{ccc} F(1) & = & 1 \\ {\scriptstyle F(\top)}\downarrow & & \downarrow{\scriptstyle a} \\ F(I) & \xrightarrow[F^*]{} & X \end{array} \qquad \begin{array}{ccc} F(1) & = & 1 \\ \downarrow{\scriptstyle F(\bot)} & & \downarrow{\scriptstyle b} \\ F(I) & \xrightarrow[F^*]{} & X \end{array}$$

And fullness and faithfulness mean that if we have any map $h : \mathrm{eval}(G) \to \mathrm{eval}(H)$, i.e.

$$h : (G(I), G(\top), G(\bot)) \to (H(I), H(\top), H(\bot))$$

in $\mathrm{Int}(\mathcal{D})$ for finite-product-preserving functors G, H, then there is a unique natural transformation $\theta : G \to H$ such that $\mathrm{eval}(\theta) = \theta_I = h$. The assumption that h is an interval homomorphism from $\mathrm{eval}(G)$ to $\mathrm{eval}(H)$ means that the following diagrams commute:

$$\begin{array}{ccccc} G(1) & \xrightarrow{G(\top)} & G(I) & = & G(I) \\ \| & & & & \downarrow{\scriptstyle h} \\ 1 & = & H(1) & \xrightarrow[H(\top)]{} & H(I) \end{array} \qquad \begin{array}{ccccc} G(1) & \xrightarrow{G(\bot)} & G(I) & = & G(I) \\ \| & & & & \downarrow{\scriptstyle h} \\ 1 & = & H(1) & \xrightarrow[H(\bot)]{} & H(I) \end{array}$$

So we can now state the definition of being the free finite-product category on an interval in the following more explicit form:

Proposition 4.2. *A category $\mathcal{C}$ is the* free finite-product category on an interval *if*

- *$\mathcal{C}$ has finite products (with terminal object 1);*
- *$\mathcal{C}$ has a distinguished interval $(I, \top, \bot)$;*

and for any category $\mathcal{D}$ with finite products (including terminal object 1) and interval (X, a, b):

1. *There is a finite-product-preserving functor $F : \mathcal{C} \to \mathcal{D}$ such that*
   - *$F(I) \cong X$ via a map $F^* : F(I) \to X$;*
   - *$F(1) \cong 1$ via a map $F^\# : F(1) \to 1$;*

   and such that both of the following equalities hold:

   $$F^* \circ F(\top) = a \circ F^\# : F(1) \to X,$$
   $$F^* \circ F(\bot) = b \circ F^\# : F(1) \to X,$$

   as in the following commutative diagrams:

   $$\begin{array}{ccc} F(1) & \xrightarrow{F^\#} & 1 \\ {\scriptstyle F(\top)}\downarrow & & \downarrow{\scriptstyle a} \\ F(I) & \xrightarrow[F^*]{} & X \end{array} \qquad \begin{array}{ccc} F(1) & \xrightarrow{F^\#} & 1 \\ \downarrow{\scriptstyle F(\bot)} & & \downarrow{\scriptstyle b} \\ F(I) & \xrightarrow[F^*]{} & X \end{array}$$

2. *Given finite-product-preserving functors $G, H : \mathcal{C} \to \mathcal{D}$ with isomorphisms*
   - *$G^* : G(I) \cong X$ and $G^\# : G(1) \cong 1$;*
   - *$H^* : H(I) \cong X$ and $H^\# : H(1) \cong 1$;*

   and given any map $h : G(I) \to H(I)$ in $\mathcal{D}$ such that both of the following equalities hold:

   $$H(\top) \circ (H^\#)^{-1} \circ G^\# = h \circ G(\top) : G(1) \to H(I),$$
   $$H(\bot) \circ (H^\#)^{-1} \circ G^\# = h \circ G(\bot) : G(1) \to H(I),$$

   as in the following commutative diagrams:

   $$\begin{array}{ccccc} G(1) & \xrightarrow{G(\top)} & G(I) & = & G(I) \\ {\scriptstyle G^\#}\downarrow & & & & {\scriptstyle h}\downarrow \\ 1 & \xrightarrow[(H^\#)^{-1}]{} & H(1) & \xrightarrow[H(\top)]{} & H(I) \end{array} \qquad \begin{array}{ccccc} G(1) & \xrightarrow{G(\bot)} & G(I) & = & G(I) \\ \downarrow{\scriptstyle G^\#} & & & & \downarrow{\scriptstyle h} \\ 1 & \xrightarrow[(H^\#)^{-1}]{} & H(1) & \xrightarrow[H(\bot)]{} & H(I) \end{array}$$

   there is a unique natural transformation $\theta : G \to H$ such that

   $$\theta_I = h : G(I) \to H(I).$$

If $\mathcal{C}$ is the free finite-product category on an interval, then any interval in any category with finite products is (up to isomorphism) the image of the universal interval $(I, \top, \bot)$ of $\mathcal{C}$ in an essentially unique way, and any interval homomorphism in a category with finite products is similarly (up to isomorphism) the component at I of a unique natural transformation.

Before we present the dual notion of being the free finite-coproduct category on a co-interval, we first make a few more preliminary definitions. The dual notion of *co-interval* (X', a', b') in a category $\mathcal{D}$ with initial object 0 is an object $X' \in \mathcal{D}$ together with two *co-points* $a', b' : X' \to 0$ from X' to the initial object. Now let $\mathcal{C}$ be any category with finite coproducts and a distinguished co-interval $(I, \top, \bot)$, so that $\top, \bot : I \to 0$. For any category $\mathcal{D}$ with finite coproducts, we define the category $\mathrm{CoInt}(\mathcal{D})$ of co-intervals of D in the obvious way, as in the finite-product case.

Also, for any category $\mathcal{D}$ with finite coproducts, we have the category $\mathrm{FCP}(\mathcal{C}, \mathcal{D})$ of finite-coproduct-preserving functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations. Finally, we have the canonical evaluation functor

$$\mathrm{eval} : \mathrm{FCP}(\mathcal{C}, \mathcal{D}) \to \mathrm{CoInt}(\mathcal{D})$$

defined as in the finite-product case. Now we can define what it means for a category $\mathcal{C}$ to be the free finite-coproduct category on a co-interval:

Definition 4.3. A category $\mathcal{C}$ is the *free finite-coproduct category on a co-interval* if $\mathcal{C}$ has finite coproducts and a distinguished co-interval $(I, \top, \bot)$ so that for any category $\mathcal{D}$ with finite coproducts, the canonical evaluation functor

$$\mathrm{eval} : \mathrm{FCP}(\mathcal{C}, \mathcal{D}) \to \mathrm{CoInt}(\mathcal{D})$$

of evaluation at the distinguished co-interval $(I, \top, \bot)$ of $\mathcal{C}$ is an equivalence of categories.

This equivalence of categories will be natural in $\mathcal{D}$, in the sense of the finite-product case.

Also as in the finite-product case, we can state the definition of being the free finite-coproduct category on a co-interval in the following more explicit form:

Proposition 4.4. *A category $\mathcal{C}$ is the* free finite-coproduct category on a co-interval *if*

- *$\mathcal{C}$ has finite coproducts (with initial object 0);*
- *$\mathcal{C}$ has a distinguished co-interval $(I, \top, \bot)$;*

and for any category $\mathcal{D}$ with finite coproducts (including initial object 0) and co-interval (X, a, b):

1. *There is a finite-coproduct-preserving functor $F : \mathcal{C} \to \mathcal{D}$ such that*

- *$F(I) \cong X$ via a map $F^* : F(I) \to X$;*
- *$F(0) \cong 0$ via some map $F^\# : F(0) \to 0$;*
- *Both of the following equalities hold:*

$$F^\# \circ F(\top) = a \circ F^* : F(I) \to 0,$$

$$F^\# \circ F(\bot) = b \circ F^* : F(I) \to 0,$$

as in the following commutative diagrams:

$$\begin{array}{ccc} F(I) & \xrightarrow{F^*} & X \\ {\scriptstyle F(\top)}\downarrow & & \downarrow{\scriptstyle a} \\ F(0) & \xrightarrow[F^\#]{} & 0 \end{array} \qquad \begin{array}{ccc} F(I) & \xrightarrow{F^*} & X \\ \downarrow{\scriptstyle F(\bot)} & & \downarrow{\scriptstyle b} \\ F(0) & \xrightarrow[F^\#]{} & 0 \end{array}$$

2. *Given finite-coproduct-preserving functors $G, H : \mathcal{C} \to \mathcal{D}$ with isomorphisms*

- *$G^* : G(I) \cong X$ and $G^\# : G(0) \cong 0$,*
- *$H^* : H(I) \cong X$ and $H^\# : H(0) \cong 0$,*

and given any map $h : G(I) \to H(I)$ such that both of the following equalities hold:

$$(H^\#)^{-1} \circ G^\# \circ G(\top) = H(\top) \circ h : G(I) \to H(0),$$
$$(H^\#)^{-1} \circ G^\# \circ G(\bot) = H(\bot) \circ h : G(I) \to H(0),$$

as in the following commutative diagrams:

$$\begin{array}{ccccc} G(I) & \xrightarrow{h} & H(I) & = & H(I) \\ {\scriptstyle G(\top)}\downarrow & & & & {\scriptstyle H(\top)}\downarrow \\ G(0) & \xrightarrow[G^\#]{} & 0 & \xrightarrow[(H^\#)^{-1}]{} & H(0) \end{array} \qquad \begin{array}{ccccc} G(I) & \xrightarrow{h} & H(I) & = & H(I) \\ \downarrow{\scriptstyle G(\bot)} & & & & \downarrow{\scriptstyle H(\bot)} \\ G(0) & \xrightarrow[G^\#]{} & 0 & \xrightarrow[(H^\#)^{-1}]{} & H(0) \end{array}$$

there is a unique natural transformation $\theta : G \to H$ such that

$$\theta_I = h : G(I) \to H(I).$$

So if $\mathcal{C}$ is the free finite-coproduct category on a co-interval, then any co-interval in any category with finite coproducts is (up to isomorphism) the image of the universal co-interval $(I, \top, \bot)$ of $\mathcal{C}$ in an essentially unique way, and any co-interval homomorphism in a category with finite coproducts is similarly (up to isomorphism) the component at I of a unique natural transformation.

To prove that $\mathcal{B}$ is the free finite-coproduct category on a co-interval, we will split the proof into two parts, proving first that $\mathcal{B}$ has finite coproducts and a distinguished co-interval and satisfies part (1) of Proposition 4.4, and then that $\mathcal{B}$ satisfies part (2) of Proposition 4.4.

Proposition 4.5. *$\mathcal{B}$ has finite coproducts and a distinguished co-interval $(I, \top, \bot)$, and satisfies part (1) of Proposition 4.4.*

Proof. First, as was noted at the end of section 3.2 in Chapter 3, it is easily seen that $\mathcal{B}$ has finite coproducts: the initial object is clearly $[0] = (\{-, +\}, -, +)$, while for any $n, m \geq 0$, we have $[n]+[m] = [n+m]$. Now we define the distinguished co-interval $(I, \top, \bot)$ of $\mathcal{B}$ as follows: we let $I = [1]$, and we define the maps $\top, \bot : [1] \to [0]$ in the obvious way to be those basepoint-preserving functions such that

$$\top(1) = +,$$

$$\bot(1) = -.$$

Now we prove that $\mathcal{B}$ satisfies part (1) of Proposition 4.4. So let $\mathcal{D}$ be any category with finite coproducts and a co-interval (X, a, b) with $a, b : X \to 0$, where we construct the desired functor $F : \mathcal{B} \to \mathcal{D}$ as follows. For any object $[m] \in \mathcal{B}$ (with $m \geq 0$), we let

$$F([m]) = X \cdot m,$$

i.e. $F([m])$ is the m^{th} copower of X. In particular, we have

$$F([0]) = X \cdot 0 \cong 0.$$

Now we define F on morphisms as follows: letting $f : [n] \to [m]$ be a map in $\mathcal{B}$, we define

$$F(f) : X \cdot n \to X \cdot m$$

as follows. Since $F(f)$ is to be a map with coproduct as domain, it suffices to define

$$F(f) \circ \text{in}_j : X \to X \cdot m$$

for each injection map $\text{in}_j : X \to X \cdot n$, where $1 \leq j \leq n$. So we define $F(f) \circ \text{in}_j$ as follows:

- If $f(j) = +$, then we let

  $$F(f) \circ \text{in}_j = ! \circ a : X \to 0 \to X \cdot m,$$

  and similarly if $f(j) = -$, then we let

  $$F(f) \circ \text{in}_j = ! \circ b : X \to 0 \to X \cdot m,$$

  where $! : 0 \to X \cdot m$ is the unique map from the initial object to $X \cdot m$.

- If $f(j) = k$ for some $1 \leq k \leq m$, then we let

  $$F(f) \circ \text{in}_j = \text{in}_k : X \to X \cdot m.$$

Now we show that $F : \mathcal{B} \to \mathcal{D}$ is functorial. First let $\mathrm{id} : [n] \to [n]$ be an identity map in $\mathcal{B}$, where we show that

$$F(\mathrm{id}) = \mathrm{id} : X \cdot n \to X \cdot n.$$

To show this, it suffices to show that for any $1 \leq j \leq n$,

$$F(\mathrm{id}) \circ \mathrm{in}_j = \mathrm{in}_j : X \to X \cdot n.$$

By definition of F on morphisms, since $\mathrm{id}(j) = j$ for any $1 \leq j \leq n$, this desired equality holds, and so $F(\mathrm{id}) = \mathrm{id}$, as desired.

Now let $f : [n] \to [m]$ and $g : [m] \to [p]$ be maps in $\mathcal{B}$, where we show that

$$F(g \circ f) = F(g) \circ F(f) : X \cdot n \to X \cdot p.$$

To show this, it suffices to show that for any $1 \leq j \leq n$,

$$F(g) \circ F(f) \circ \mathrm{in}_j = F(g \circ f) \circ \mathrm{in}_j : X \to X \cdot p.$$

- First suppose that $g(f(j)) = +$, because $f(j) = +$. Then we have

  $$\begin{aligned} F(g) \circ F(f) \circ \mathrm{in}_j &= F(g) \circ ! \circ a \\ &= ! \circ a \\ &= F(g \circ f) \circ \mathrm{in}_j, \end{aligned}$$

  as desired.

  Now suppose that $g(f(j)) = +$ because $f(j) = k$ for some $1 \leq k \leq m$ and $g(k) = +$. Then we have

  $$\begin{aligned} F(g) \circ F(f) \circ \mathrm{in}_j &= F(g) \circ \mathrm{in}_k \\ &= ! \circ a \\ &= F(g \circ f) \circ \mathrm{in}_j, \end{aligned}$$

  as desired.

  So in any case where $g(f(j)) = +$, we have $F(g) \circ F(f) \circ \mathrm{in}_j = F(g \circ f) \circ \mathrm{in}_j$, as desired.

- Similarly, if $g(f(j)) = -$, then in either possible case, we have

  $$F(g) \circ F(f) \circ \mathrm{in}_j = F(g \circ f) \circ \mathrm{in}_j = ! \circ b : X \to X \cdot p,$$

  as desired.

- Lastly, suppose that $g(f(j)) = q$ for some $1 \leq q \leq p$, so that we must have $f(j) = k$ for some $1 \leq k \leq n$ and $g(k) = q$. Then we have

$$\begin{aligned} F(g) \circ F(f) \circ \mathrm{in}_j &= F(g) \circ \mathrm{in}_k \\ &= \mathrm{in}_q \\ &= F(g \circ f) \circ \mathrm{in}_j, \end{aligned}$$

  as desired.

So in any possible case on the value of $g(f(j))$, we have $F(g) \circ F(f) \circ \mathrm{in}_j = F(g \circ f) \circ \mathrm{in}_j$, as desired, whereby $F(g) \circ F(f) = F(g \circ f)$, so that F is functorial.

Now, the definition of F easily implies that F preserves finite coproducts, and moreover we obviously have the isomorphisms

$$\mathrm{id} : F([1]) = X \cong X,$$

$$\mathrm{id} : F([0]) = X \cdot 0 \cong 0,$$

as required. So now it only remains to show that the diagrams in part (1) of Proposition 4.4 commute. Well, it follows by definition of F on morphisms that $F(\top) = a : X \to 0$ and $F(\bot) = b : X \to 0$, and so the relevant diagrams obviously commute, as required. This completes the proof that $\mathcal{B}$ satisfies part (1) of Proposition 4.4.

□

Now we prove that $F : \mathcal{B} \to \mathcal{D}$ satisfies part (2) of Proposition 4.4.

Proposition 4.6. *$F : \mathcal{B} \to \mathcal{D}$ satisfies part (2) of Proposition 4.4.*

Proof. Suppose that we have finite-coproduct-preserving functors $G, H : \mathcal{B} \to \mathcal{D}$ satisfying the assumptions in part (2) of Proposition 4.4, i.e. we have isomorphisms

- $G^* : G([1]) \cong X$ and $G^\# : G([0]) \cong 0$,
- $H^* : H([1]) \cong X$ and $H^\# : H([0]) \cong 0$.

And suppose that we have a map $h : G(I) \to H(I)$ such that both of the following equalities hold:

$$(H^\#)^{-1} \circ G^\# \circ G(\top) = H(\top) \circ h : G([1]) \to H([0]),$$

$$(H^\#)^{-1} \circ G^\# \circ G(\bot) = H(\bot) \circ h : G([1]) \to H([0]).$$

Then we want to show that there is a unique natural transformation $\theta : G \to H$ such that

$$\theta_{[1]} = h : G([1]) \to H([1]).$$

We define a natural transformation $\theta : G \to H$ as follows. First, we let

$$\theta_{[0]} = (H^{\#})^{-1} \circ G^{\#} : G([0]) \to 0 \to H([0]),$$

and we let

$$\theta_{[1]} = h : G([1]) \to H([1]).$$

Now (for $n \geq 2$), consider $[n] = [1] \cdot n$. Since G and H are finite-coproduct-preserving, we have the following two isomorphisms in $\mathcal{D}$:

$$G^n : G([1] \cdot n) \cong G([1]) \cdot n,$$

$$H^n : H([1] \cdot n) \cong H([1] \cdot n.$$

Also, we have $h : G([1]) \to H([1])$, and hence we have

$$[\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] : G([1]) \cdot n \to H([1]) \cdot n,$$

where $\mathrm{in}_{H,j} : H([1]) \to H([1]) \cdot n$ for $1 \leq j \leq n$ is an injection map. Finally, we let

$$\theta_{[n]} = (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n$$

$$: G([n]) = G([1] \cdot n) \to H([1] \cdot n) = H([n]),$$

as in the following commutative diagram:

$$\begin{array}{ccc} G([n]) & \xrightarrow{\theta_{[n]}} & H([n]) \\ {\scriptstyle G^n}\downarrow & & \uparrow{\scriptstyle (H^{\#})^{-1}} \\ G([1] \cdot n) & \xrightarrow[{[\mathrm{in}_1 \circ h, \ldots, \mathrm{in}_n \circ h]}]{} & H([1] \cdot n) \end{array}$$

This completes the definition of $\theta : G \to H$.

Now we must show that θ is natural. First consider any map $! : [0] \to [m]$ in $\mathcal{B}$ for any $m \geq 0$. Then we automatically have

$$\theta_{[m]} \circ G(!) = H(!) \circ \theta_{[0]} : G([0]) \to H([m]),$$

because these are both parallel maps in $\mathcal{D}$ from the initial object $G([0])$ of $\mathcal{D}$.

Now we consider maps from $[1]$ to $[0]$ in $\mathcal{B}$, where $\top, \bot : [1] \to [0]$ are the only two such maps in $\mathcal{B}$. So we must show that both of the following equalities hold:

$$\theta_{[0]} \circ G(\top) = H(\top) \circ \theta_{[1]},$$

$$\theta_{[0]} \circ G(\bot) = H(\bot) \circ \theta_{[1]},$$

as in the following diagrams:

$$\begin{array}{ccc} G([1]) & \xrightarrow{\theta_{[1]}} & H([1]) \\ {\scriptstyle G(\top)}\downarrow & & \downarrow{\scriptstyle H(\top)} \\ G([0]) & \xrightarrow[\theta_{[0]}]{} & H([0]) \end{array} \qquad \begin{array}{ccc} G([1]) & \xrightarrow{\theta_{[1]}} & H([1]) \\ \downarrow{\scriptstyle G(\bot)} & & \downarrow{\scriptstyle H(\bot)} \\ G([0]) & \xrightarrow[\theta_{[0]}]{} & H([0]) \end{array}$$

But by definition of θ, we have $\theta_{[0]} = (H^{\#})^{-1} \circ G^{\#}$ and $\theta_{[1]} = h$, so we are reduced to showing that the following equalities both hold:

$$(H^{\#})^{-1} \circ G^{\#} \circ G(\top) = H(\top) \circ h,$$

$$(H^{\#})^{-1} \circ G^{\#} \circ G(\bot) = H(\bot) \circ h,$$

but these equalities follow from the assumptions about $h : G([1]) \to H([1])$.

Now consider any map $f : [1] \to [n]$ in $\mathcal{B}$ for any $n \geq 1$, where we must show that

$$\theta_{[n]} \circ G(f) = H(f) \circ \theta_{[1]},$$

as in the following diagram:

$$\begin{array}{ccc} G([1]) & \xrightarrow{\theta_{[1]}} & H([1]) \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle H(f)} \\ G([n]) & \xrightarrow[\theta_{[n]}]{} & H([n]) \end{array}$$

By definition of $\theta_{[n]}$ and $\theta_{[1]}$, we must actually show that

$$(H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \circ G(f) = H(f) \circ h.$$

We know that either $f(1) = +$, $f(1) = -$, or $f(1) = j$ for some $1 \leq j \leq n$. First suppose that f maps 1 to one of the basepoints: we consider only the case where $f(1) = +$, since the other case is similar. So we have that

$$f = ! \circ \top : [1] \to [0] \to [n],$$

so that

$$G(f) = G(!) \circ G(\top),$$

$$H(f) = H(!) \circ H(\top).$$

So we must show

$$(H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \circ G(!) \circ G(\top) = H(!) \circ H(\top) \circ h,$$

so we calculate as follows:

$$\begin{aligned} (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \circ G(!) \circ G(\top) &= (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \\ &\quad \circ ! \circ G^{\#} \circ G(\top) \\ &= ! \circ G^{\#} \circ G(\top) \\ &= ! \circ H^{\#} \circ (H^{\#})^{-1} \circ G^{\#} \circ G(\top) \\ &= ! \circ H^{\#} \circ H(\top) \circ h \\ &= H(!) \circ H(\top) \circ h. \end{aligned}$$

Now assume that $f(1) = j$ for some $1 \leq j \leq n$, which entails that

$$f = \mathrm{in}_j : [1] \to [n].$$

So we must show

$$(H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \circ G(\mathrm{in}_j) = H(\mathrm{in}_j) \circ h.$$

This equality follows just because G and H preserve finite coproducts. Explicitly, we have both

$$\begin{aligned} H^n \circ H(\mathrm{in}_j) &: H([1]) \to H([1] \cdot n) \to H([1]) \cdot n, \\ G^n \circ G(\mathrm{in}_j) &: G([1]) \to G([1] \cdot n) \to G([1]) \cdot n. \end{aligned}$$

So because G and H preserve finite coproducts, we obtain both

$$H^n \circ H(\mathrm{in}_j) = \mathrm{in}_{H,j} : H([1]) \to H([1]) \cdot n,$$

$$G^n \circ G(\mathrm{in}_j) = \mathrm{in}_{G,j} : G([1]) \to G([1]) \cdot n.$$

Then we have

$$\begin{aligned} [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \circ G(\mathrm{in}_j) &= [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ \mathrm{in}_{G,j} \\ &= \mathrm{in}_{H,j} \circ h \\ &= H^n \circ H(\mathrm{in}_j) \circ h, \end{aligned}$$

from which the desired naturality equation follows.

Finally, consider any map $f : [1] \cdot n = [n] \to [m]$ in $\mathcal{B}$ for some $n \geq 2$ and $m \geq 1$, where we must show that

$$\theta_{[m]} \circ G(f) = H(f) \circ \theta_{[n]},$$

as in the following diagram:

$$\begin{array}{ccc} G([n]) & \xrightarrow{\theta_{[n]}} & H([n]) \\ {\scriptstyle G(f)}\downarrow & & \downarrow{\scriptstyle H(f)} \\ G([m]) & \xrightarrow[\theta_{[m]}]{} & H([m]) \end{array}$$

Well, we have that $f = [f_1, \ldots, f_n]$, where $f_j : [1] \to [n]$ for each $1 \leq j \leq n$, where we have already shown that

$$\theta_{[m]} \circ G(f_j) = H(f_j) \circ h$$

for each such j. In other words, for each $1 \leq j \leq n$ we already know that

$$[\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \circ G^m \circ G(f_j) = H^m \circ H(f_j) \circ h.$$

So what we must now show is

$$\theta_{[m]} \circ G([f_1, \ldots, f_n]) = H([f_1, \ldots, f_n]) \circ \theta_{[n]};$$

so we calculate as follows:

$$\begin{aligned}
H^m \circ \theta_{[m]} \circ G([f_1, \ldots, f_n]) \circ (G^n)^{-1} &= [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \circ G^m \circ G([f_1, \ldots, f_n]) \circ (G^n)^{-1} \\
&= [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \circ [G^m \circ G(f_1), \ldots, G^m \circ G(f_n)] \\
&= [[\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \circ G^m \circ G(f_1), \ldots, \\
&\quad [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \circ G^m \circ G(f_n)] \\
&= [H^m \circ H(f_1) \circ h, \ldots, H^m \circ H(f_n) \circ h] \\
&= [[H^m \circ H(f_1), \ldots, H^m \circ H(f_n)] \circ \mathrm{in}_{H,1} \circ h, \ldots, \\
&\quad [H^m \circ H(f_1), \ldots, H^m \circ H(f_n)] \circ \mathrm{in}_{H,n} \circ h] \\
&= [H^m \circ H(f_1), \ldots, H^m \circ H(f_n)] \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \\
&= H^m \circ H([f_1, \ldots, f_n]) \circ (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h].
\end{aligned}$$

Therefore, we obtain

$$(H^m)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \circ (G^m) \circ G([f_1, \ldots, f_n])$$

$$= H([f_1, \ldots, f_n]) \circ (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n,$$

i.e. we have

$$\theta_{[m]} \circ G([f_1, \ldots, f_n]) = H([f_1, \ldots, f_n]) \circ \theta_{[n]},$$

as desired. This completes the proof that $\theta : G \to H$ is a natural transformation such that $\theta_{[1]} = h : G([1]) \to H([1])$.

Now we must show that $\theta : G \to H$ is the unique natural transformation such that $\theta_{[1]} = h$. So let $\eta : G \to H$ be any natural transformation such that $\eta_{[1]} = h$, where we must show that $\theta = \eta$. First we show that

$$\theta_{[0]} = (H^{\#})^{-1} \circ G^{\#} = \eta_{[0]}.$$

Since η is natural and $\eta_{[1]} = h$, we have

$$\eta_{[0]} \circ G(\top) = H(\top) \circ h$$

and we also have

$$(H^{\#})^{-1} \circ G^{\#} \circ G(\top) = H(\top) \circ h.$$

Therefore, we have

$$\eta_{[0]} \circ G(\top) = (H^{\#})^{-1} \circ G^{\#} \circ G(\top).$$

But since $G(\top) : G([1]) \to G([0])$ and $G([0])$ is initial in $\mathcal{D}$, it follows that $G(\top)$ is epi, from which the desired equality follows:

$$\eta_{[0]} = (H^{\#}) \circ G^{\#} = \theta_{[0]}.$$

We obviously have
$$\eta_{[1]} = h = \theta_{[1]}.$$
Now consider $[n] = 1 \cdot n$ for $n \geq 2$, where we must show that
$$\eta_{[n]} = (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n = \theta_{[n]}.$$
Since η and θ are natural and $\eta_{[1]} = h = \theta_{[1]}$, we have both
$$(H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \circ G(\mathrm{in}_j) = H(\mathrm{in}_j) \circ h,$$
$$\eta_{[n]} \circ G(\mathrm{in}_j) = H(\mathrm{in}_j) \circ h,$$
where $\mathrm{in}_j : [1] \to [1] \cdot n = [n]$ for any $1 \leq j \leq n$. Therefore, we have
$$(H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n \circ G(\mathrm{in}_j) = \eta_{[n]} \circ G(\mathrm{in}_j),$$
i.e. we have
$$(H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ \mathrm{in}_{G,j} = \eta_{[n]} \circ G(\mathrm{in}_j).$$
So we have
$$(H^n)^{-1} \circ \mathrm{in}_{H,j} \circ h = \eta_{[n]} \circ G(\mathrm{in}_j),$$
and hence
$$H(\mathrm{in}_j) \circ h = \eta_{[n]} \circ G(\mathrm{in}_j).$$
So for any $1 \leq j \leq n$, we have
$$H(\mathrm{in}_j) \circ h = \eta_{[n]} \circ G(\mathrm{in}_j).$$

To show
$$\eta_{[n]} = (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n = \theta_{[n]},$$
it suffices to show
$$H^n \circ \eta_{[n]} \circ (G^n)^{-1} = [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h].$$
And to show *this*, it suffices to show that
$$H^n \circ \eta_{[n]} \circ (G^n)^{-1} \circ \mathrm{in}_{G,j} = \mathrm{in}_{H,j} \circ h$$
for each $1 \leq j \leq n$. So, we have
$$\begin{aligned} H^n \circ \eta_{[n]} \circ (G^n)^{-1} \circ \mathrm{in}_{G,j} &= H^n \circ \eta_{[n]} \circ G(\mathrm{in}_j) \\ &= H^n \circ H(\mathrm{in}_j) \circ h \\ &= \mathrm{in}_{H,j} \circ h. \end{aligned}$$

This completes the proof that $\eta_{[n]} = \theta_{[n]}$ for $n \geq 2$, and hence that $\eta = \theta$. Therefore, $\theta : G \to H$ is the unique natural transformation such that $\theta_{[1]} = h$. So $\mathcal{B}$ satisfies part (2) of Proposition 4.4, as was to be shown.

□

Combining Propositions 4.5 and 4.6, we now obtain:

Theorem 4.7. *$\mathcal{B}$ is the free finite-coproduct category on a co-interval.*

The duality $\mathcal{B} \cong \mathbb{C}^{op}$ established in the previous chapter now yields the fact that $\mathbb{C}$ has the dual classifying property defined initially:

Corollary 4.8. *$\mathbb{C}$ is the free finite-product category on an interval.*

As the free finite-product category on an interval, the distinguished interval of $\mathbb{C}$ consists in the 1-cube I and the two face maps $\phi_0^0, \phi_0^1 : I^0 \to I$.

4.2 Finite Weakly Bipointed Sets

In the preceding section we proved that the category $\mathcal{B}$ of finite *strictly* bipointed sets has the classifying property of being the free finite-coproduct category on a co-interval. In this section, we will introduce the category $\mathcal{B}_w$ of finite *weakly* bipointed sets, whose objects will be finite sets equipped with two basepoints that are not necessarily distinct, and whose maps will again be any functions that preserve the basepoints. We will then prove that this category $\mathcal{B}_w$ has the classifying property of being the free finite-*colimit* category on a co-interval. Here is the explicit definition of the category $\mathcal{B}_w$ of finite weakly bipointed sets:

Definition 4.9. The category $\mathcal{B}_w$ of finite weakly bipointed sets is defined as follows:

- *Objects*: For any $n \geq 0$, $\mathcal{B}_w$ contains the objects

$$[n] = (\{1, \dots, n, -, +\}, -, +),$$

$$[n]_w = (\{1, \dots, n, \#\}, \#, \#).$$

- *Maps*: As in $\mathcal{B}$, a map in $\mathcal{B}_w$ is just a function that preserves the basepoints.

Composition and identities are defined in the obvious way.

So an object of $\mathcal{B}_w$ is a finite bipointed set with two basepoints that may coincide. Note that there are no maps in $\mathcal{B}_w$ from a pointed set $[n]_w$ to a strictly bipointed set $[m]$. In preparation for showing that $\mathcal{B}_w$ is the free finite-colimit category on a co-interval, we now show that $\mathcal{B}_w$ has finite colimits (by showing that it has finite coproducts and coequalizers). First we show that $\mathcal{B}_w$ has finite coproducts.

Proposition 4.10. *$\mathcal{B}_w$ has finite coproducts, constructed as follows:*

- Initial object: $[0] = \{-, +\}$.

- Binary coproducts:

  - $[n] + [m] = [n + m]$.

– $[n]_w + [m]_w = [n+m]_w$.

– $[n]_w + [m] = [n+m]_w$.

Proof. First, $[0] = \{-, +\}$ is the initial object because there is exactly one map from $[0]$ to any other finite weakly bipointed set, namely the map that simply preserves the basepoints of $[0]$.

To show that $[n] + [m] = [n+m]$, note first that there are obvious injection maps $[n], [m] \to [n+m]$. For explicitness, we take $in_1 : [n] \to [n+m]$ to be that basepoint-preserving function such that $in_1(j) = j$ for any $1 \leq j \leq n$, and we take $in_2 : [m] \to [n+m]$ to be that basepoint-preserving function such that $in_2(k) = n+k$ for any $1 \leq k \leq m$. Now let $f : [n] \to [p]$, $g : [m] \to [p]$ be maps in $\mathcal{B}_w$ (the case where $f : [n] \to [p]_w$, $g : [m] \to [p]_w$ is similar), where we construct a unique map $[f, g] : [n+m] \to [p]$ such that the following diagram commutes:

$$\begin{array}{ccccc} [n] & \xrightarrow{in_1} & [n+m] & \xleftarrow{in_2} & [m] \\ & {\scriptstyle f}\searrow & \vdots{\scriptstyle [f,g]} & \swarrow{\scriptstyle g} & \\ & & [p] & & \end{array}$$

For any $1 \leq j \leq n$, we let $[f, g](j) = f(j)$, and for any $n+1 \leq k \leq n+m$, we let $[f, g](k) = g(k-n)$ (and we stipulate that $[f, g]$ preserves basepoints, as required). Then it is clear that $[f, g]$ is the unique map making the diagram commute. So we have shown that $[n] + [m] = [n+m]$. An exactly parallel proof shows that $[n]_w + [m]_w = [n+m]_w$.

Now we show that $[n]_w + [m] = [n+m]_w$. As before, we have obvious basepoint-preserving injection maps $in_1 : [n]_w \to [n+m]_w$ and $in_2 : [m] \to [n+m]_w$. Now let f, g be maps from the cofactors to a common codomain. Since one of the maps has the pointed set $[n]_w$ for its domain, it follows that the common codomain must also be a pointed set, say $[p]_w$. So we have $f : [n]_w \to [p]_w$ and $g : [m] \to [p]_w$. Now we must construct a unique map $[f, g] : [n+m]_w \to [p]_w$ making the following diagram commute:

$$\begin{array}{ccccc} [n]_w & \xrightarrow{in_1} & [n+m]_w & \xleftarrow{in_2} & [m] \\ & {\scriptstyle f}\searrow & \vdots{\scriptstyle [f,g]} & \swarrow{\scriptstyle g} & \\ & & [p]_w & & \end{array}$$

Defining $[f, g]$ in the same way as for the first case of the coproduct of two strictly bipointed sets gives the required unique map. So $[n]_w + [m] = [n+m]_w$.

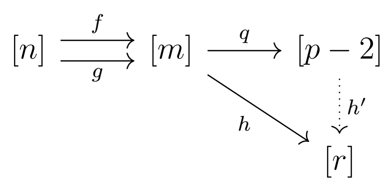
□

Now we give a detailed account of coequalizers in $\mathcal{B}_w$. First suppose that we have maps $f, g : [n] \to [m]$ between finite strictly bipointed sets. Consider the reflexive and symmetric relation R defined on $\{1, \ldots, m, -, +\}$ as follows: for any $a, b \in [m]$, aRb iff one of the following three conditions holds:

- $a = b$; or
- There is some $x \in [n]$ such that $f(x) = a$ and $g(x) = b$; or
- There is some $y \in [n]$ such that $f(y) = b$ and $g(y) = a$.

Now let R^* be the transitive closure of this reflexive and symmetric relation R, so that R^* is an equivalence relation. Let $p \geq 1$ be the finite cardinality of the set of equivalence classes of the equivalence relation R^* defined on the set $\{1, \ldots, m, -, +\}$. First suppose that $(-, +) \notin R$ (so that $p \geq 2$). Then we define a map $q : [m] \to [p-2]$ as follows. For any $1 \leq k \leq m$, if kR^*+, then we let $q(k) = +$, and similarly for $-$. Otherwise (i.e. if we have neither kR^*+ nor kR^*-), then we let $q(k) = k'$, where $1 \leq k' \leq p-2$ is the number of the equivalence class under R^* to which k belongs (where we let $p-1$ and p be the numbers of the equivalence classes of $+$ and $-$, respectively). Then it is easy to see that q coequalizes f and g.

Now let $h : [m] \to [r]$ be a map between strictly bipointed sets (the argument is the same if $[r]_w$ is only a pointed set) such that $h \circ f = h \circ g$, where we show that there is a unique map $h' : [p-2] \to [r]$ such that $h' \circ q = h$, as in the following diagram:



For any $1 \leq t \leq p-2$, we let $h'(t) = h(k)$, where t is the number of the equivalence class of k under R^* (for some $1 \leq k \leq m$). Now we must show that this definition of h' respects the equivalence relation R^*, i.e. that if $1 \leq t \leq p-2$ is such that t is the number of the equivalence class of both k and k' under R^* (for some $1 \leq k, k' \leq m$), then we have $h(k) = h(k')$. So we have kR^*k'. Then either kRk', or there are $x_1, \ldots, x_r \in [m]$ such that $kRx_1R \ldots Rx_r \ldots Rk'$ (for some $r \geq 1$). First suppose that kRk'. If this holds because $k = k'$, then clearly we have $h(k) = h(k')$. Now suppose that kRk' holds because there is some $x \in [n]$ such that $f(x) = k$ and $g(x) = k'$ (the other case is similar). Then since $h \circ f = h \circ g$, we have

$$h(k) = h(f(x)) = h(g(x)) = h(k'),$$

as desired. Now assume that kR^*k' because there are $x_1, \ldots, x_r \in [m]$ such that $kRx_1R \ldots Rx_rRk'$ for some $r \geq 1$. But as we have just seen, if $a, b \in [m]$ are such that aRb, then we have $h(a) = h(b)$. So it follows that

$$h(k) = h(x_1) = \ldots = h(x_r) = h(k'),$$

so that $h(k) = h(k')$, as desired.

So h' is well-defined, and now we show that $h' \circ q = h$. If $1 \leq k \leq m$ is not equivalent (under R^*) to either basepoint, then we have $h'(q(k)) = h(k)$, as desired. But if k is equivalent to one of the basepoints, say $+$ (the argument for $-$ is analogous), then we must show that $h(k) = +$, since

$$h'(q(k)) = h'(+) = +.$$

Well, (since $k \neq +$) suppose first that kR^*+ because there is some $x \in [n]$ such that $f(x) = k$ and $g(x) = +$ (the other case is similar). Then we have

$$h(k) = h(f(x)) = h(g(x)) = h(+) = +,$$

as desired. Finally, assume that kR^*+ because there are $x_1, \ldots, x_r \in [m]$ (for some $r \geq 1$) such that $kRx_1R \ldots Rx_rR+$. Then as we have previously observed, we obtain

$$h(k) = h(x_1) = \ldots = h(x_r) = h(+) = +,$$

again as desired.

So we have $h' \circ q = h$, and the uniqueness of h' follows because q is epi.

If we now suppose that $(-, +) \in R$, then we define a map $q : [m] \to [p-1]_w$ in the exact same way as before, from which it follows in the same way that q is the coequalizer of f and g. In the same way, the coequalizer of two maps $f, g : [n]_w \to [m]_w$ will be a map $q : [m]_w \to [p-1]_w$ defined in the same way as before, and the coequalizer of two maps $f, g : [n] \to [m]_w$ will again be a map $q : [m]_w \to [p-1]_w$ defined in the same way as before.

In summary, $\mathcal{B}_w$ has finite coproducts and coequalizers, and hence has finite colimits. Now we present the definition of being the free finite-colimit category on a co-interval, before we show that $\mathcal{B}_w$ has this classifying property.

Definition 4.11. A category $\mathcal{C}$ is the *free finite-colimit category on a co-interval* if $\mathcal{C}$ has finite colimits and a distinguished co-interval $(I, \top, \bot)$ so that for any category $\mathcal{D}$ with finite colimits, the canonical evaluation functor

$$\text{eval} : \text{FCLP}(\mathcal{C}, \mathcal{D}) \to \text{CoInt}(\mathcal{D})$$

of evaluation at the distinguished co-interval $(I, \top, \bot)$ of $\mathcal{C}$ is an equivalence of categories.

Here FCLP($\mathcal{C}$, $\mathcal{D}$) is the category of finite-colimit-preserving functors from $\mathcal{C}$ to $\mathcal{D}$ and natural transformations, while eval : $\text{FCLP}(\mathcal{C}, \mathcal{D}) \to \text{CoInt}(\mathcal{D})$ is defined as it was in the case of the free finite-coproduct category on a co-interval.

As before, this equivalence of categories will be natural in $\mathcal{D}$, in the sense that for any finite-colimit-preserving functor $H : \mathcal{D} \to \mathcal{D}'$ (where $\mathcal{D}'$ is a category with finite colimits), the following diagram commutes:

$$\begin{array}{ccc} \text{FCLP}(\mathcal{C}, \mathcal{D}) & \xrightarrow{\text{eval}} & \text{CoInt}(\mathcal{D}) \\ {\scriptstyle \text{FCLP}(\mathcal{C},H)} \downarrow & & \downarrow {\scriptstyle \text{CoInt}(H)} \\ \text{FCLP}(\mathcal{C}, \mathcal{D}') & \xrightarrow[\text{eval}]{} & \text{CoInt}(\mathcal{D}'), \end{array}$$

where

$$\mathrm{FCLP}(\mathcal{C}, H) : \mathrm{FCLP}(\mathcal{C}, \mathcal{D}) \to \mathrm{FCLP}(\mathcal{C}, \mathcal{D}')$$

and

$$\mathrm{CoInt}(H) : \mathrm{CoInt}(\mathcal{D}) \to \mathrm{CoInt}(\mathcal{D}')$$

are defined as before.

As before, we can present the following more explicit version of Definition 4.11 as follows:

Proposition 4.12. *A category $\mathcal{C}$ is the* free finite-colimit category on a co-interval *if*

- *$\mathcal{C}$ has finite colimits (with initial object 0);*
- *$\mathcal{C}$ has a distinguished co-interval $(I, \top, \bot)$;*

and for any category $\mathcal{D}$ with finite colimits (including initial object 0) and co-interval (X, a, b):

1. *There is a finite-colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}$ such that*
   - *$F(I) \cong X$ via a map $F^* : F(I) \to X$;*
   - *$F(0) \cong 0$ via some map $F^\# : F(0) \to 0$;*
   - *Both of the following equalities hold:*

   $$F^\# \circ F(\top) = a \circ F^* : F(I) \to 0,$$

   $$F^\# \circ F(\bot) = b \circ F^* : F(I) \to 0,$$

   as in the following commutative diagrams:

   $$\begin{array}{ccc} F(I) & \xrightarrow{F^*} & X \\ {\scriptstyle F(\top)}\downarrow & & \downarrow{\scriptstyle a} \\ F(0) & \xrightarrow[F^\#]{} & 0 \end{array} \qquad \begin{array}{ccc} F(I) & \xrightarrow{F^*} & X \\ \downarrow{\scriptstyle F(\bot)} & & \downarrow{\scriptstyle b} \\ F(0) & \xrightarrow[F^\#]{} & 0 \end{array}$$

2. *Given finite-colimit-preserving functors $G, H : \mathcal{C} \to \mathcal{D}$ with isomorphisms*
   - *$G^* : G(I) \cong X$ and $G^\# : G(0) \cong 0$,*
   - *$H^* : H(I) \cong X$ and $H^\# : H(0) \cong 0$,*

   and given any map $h : G(I) \to H(I)$ such that both of the following equalities hold:

   $$(H^\#)^{-1} \circ G^\# \circ G(\top) = H(\top) \circ h : G(I) \to H(0),$$
   $$(H^\#)^{-1} \circ G^\# \circ G(\bot) = H(\bot) \circ h : G(I) \to H(0),$$

   as in the following commutative diagrams:

$$\begin{array}{ccccccccccc} G(I) & \xrightarrow{h} & H(I) & = & H(I) & & G(I) & \xrightarrow{h} & H(I) & = & H(I) \\ \downarrow{\scriptstyle G(\top)} & & & & \downarrow{\scriptstyle H(\top)} & & \downarrow{\scriptstyle G(\bot)} & & & & \downarrow{\scriptstyle H(\bot)} \\ G(0) & \xrightarrow[G^{\#}]{} & 0 & \xrightarrow[(H^{\#})^{-1}]{} & H(0) & & G(0) & \xrightarrow[G^{\#}]{} & 0 & \xrightarrow[(H^{\#})^{-1}]{} & H(0) \end{array}$$

there is a unique natural transformation $\theta : G \to H$ such that

$$\theta_I = h : G(I) \to H(I).$$

So, as before, if $\mathcal{C}$ is the free finite-colimit category on a co-interval, then any co-interval in any category with finite colimits is (up to isomorphism) the image of the universal co-interval $(I, \top, \bot)$ of $\mathcal{C}$ in an essentially unique way, and any co-interval homomorphism in a category with finite colimits is similarly (up to isomorphism) the component at I of a unique natural transformation.

As when we proved that $\mathcal{B}$ is the free finite-coproduct category on a co-interval, we will prove that $\mathcal{B}_w$ is the free finite-colimit category on a co-interval by showing first that $\mathcal{B}_w$ satisfies part (1) of Proposition 4.12, and then showing that $\mathcal{B}_w$ satisfies part (2). As we just showed, $\mathcal{B}_w$ has finite colimits, as required, and the distinguished co-interval of $\mathcal{B}_w$ is the same as that for $\mathcal{B}$, namely the triple $([1], \top, \bot)$ with $\top, \bot : [1] \to [0]$ such that

$$\top(1) = +,$$
$$\bot(1) = -.$$

Proposition 4.13. *$\mathcal{B}_w$ satisfies part (1) of Proposition 4.12.*

Proof. Let $\mathcal{D}$ be any category with finite colimits and co-interval (X, a, b), so that $a, b : X \to 0$. We must construct a finite-colimit-preserving functor $F : \mathcal{B}_w \to \mathcal{D}$ such that

- $F(I) \cong X$ via a map $F^* : F(I) \to X$;
- $F(0) \cong 0$ via some map $F^{\#} : F(0) \to 0$;
- Both of the following equalities hold:

$$F^{\#} \circ F(\top) = a \circ F^* : F(I) \to 0,$$

$$F^{\#} \circ F(\bot) = b \circ F^* : F(I) \to 0.$$

We begin defining the desired functor $F : \mathcal{B}_w \to \mathcal{D}$ on objects as follows: for any $n \geq 0$ and any strictly bipointed set $[n]$, we let

$$F([n]) = X \cdot n,$$

and for any $n \geq 0$ and any pointed set $[n]_w$, we let

$$F([n]_w) = |\mathrm{Coker}(!_n \circ a, !_n \circ b)|,$$

i.e. the codomain of the coequalizer map

$$\mathrm{Coker}(!_n \circ a, !_n \circ b) : X \cdot n \to |\mathrm{Coker}(!_n \circ a, !_n \circ b)|,$$

where $!_n : 0 \to X \cdot n$, so that

$$!_n \circ a, !_n \circ b : X \to X \cdot n,$$

all as in the following diagram:

$$X \underset{!_n \circ b}{\overset{!_n \circ a}{\rightrightarrows}} X \cdot n \xrightarrow{\mathrm{Coker}} |\mathrm{Coker}(!_n \circ a, !_n \circ b)|$$

For readability, we will from now on write

$$Q_n \doteq F([n]_w) = |\mathrm{Coker}(!_n \circ a, !_n \circ b)|,$$

$$q_n \doteq \mathrm{Coker}(!_n \circ a, !_n \circ b) : X \cdot n \to Q_n.$$

Now we define F on morphisms as follows.

1. First let $f : [n] \to [m]$ be a map between strictly bipointed sets, where we define
$$F(f) : X \cdot n \to X \cdot m$$
exactly as in the proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval. Namely, since the domain of the desired map $F(f)$ is a coproduct, it suffices to define
$$F(f) \circ \mathrm{in}_j : X \to X \cdot m$$
for any $1 \leq j \leq n$.
    - If $f(j) = +$, we let
    $$F(f) \circ \mathrm{in}_j = !_m \circ a : X \to 0 \to X \cdot m.$$
    - Similarly, if $f(j) = -$, we let
    $$F(f) \circ \mathrm{in}_j = !_m \circ b : X \to 0 \to X \cdot m.$$
    - If $f(j) = k$ for some $1 \leq k \leq m$, then we let
    $$F(f) \circ \mathrm{in}_j = \mathrm{in}_k : X \to X \cdot m.$$
2. Now let $f : [n]_w \to [m]_w$ be a map in $\mathcal{B}_w$ between two pointed sets, where we must define
$$F(f) : Q_n \to Q_m.$$
Now, it is readily seen that in $\mathcal{B}_w$,
$$[n]_w = |\mathrm{Coker}(+_n, -_n)|,$$

where

$$+_n, -_n : [1] \to [n]$$

are such that

$$+_n(1) = +,$$

$$-_n(1) = -.$$

So given f, we then have

$$f \circ \mathrm{Coker}(+_n, -_n) : [n] \to [n]_w \to [m]_w.$$

Now we find a lift

$$f' : [n] \to [m]$$

of $f \circ \mathrm{Coker}(+_n, -_n)$, i.e. a map f' such that

$$\mathrm{Coker}(+_m, -_m) \circ f' = f \circ \mathrm{Coker}(+_n, -_n) : [n]_w \to [m]_w,$$

as in the following diagram:

$$\begin{array}{ccc} [n] & \xrightarrow{\mathrm{Coker}(+_n,-_n)} & [n]_w \\ {\scriptstyle f'} \downarrow & & \downarrow {\scriptstyle f} \\ [m] & \xrightarrow{\mathrm{Coker}(+_m,-_m)} & [m]_w \end{array}$$

So let $1 \leq j \leq n$, where we define $f'(j) \in [m]$ as follows:

- If $f(\mathrm{Coker}(+_n, -_n))(j) = k$, for some $1 \leq k \leq m$, then we let

$$f'(j) = k.$$

- If $f(\mathrm{Coker}(+_n, -_n))(j) = \#$, then we let

$$f'(j) = -.$$

Now it easily follows that

$$\mathrm{Coker}(+_m, -_m) \circ f' = f \circ \mathrm{Coker}(+_n, -_n) : [n]_w \to [m]_w.$$

Note also that we have both

$$f' \circ +_n = +_m : [1] \to [m],$$

$$f' \circ -_n = -_m : [1] \to [n],$$

because all of these maps preserve basepoints (as maps of $\mathcal{B}_w$). Since

$$F(+_n) = !_n \circ a : X \to X \cdot n,$$

$$F(-_n) = !_n \circ b : X \to X \cdot n,$$

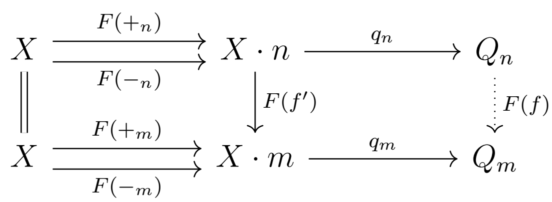
and likewise for $+_m, -_m$, it follows that we have

$$\begin{aligned} F(f') \circ F(+_n) &= F(f') \circ !_n \circ a \\ &= F(+_m) \\ &= !_m \circ a, \end{aligned}$$

and likewise we have

$$F(f') \circ F(-_n) = !_m \circ b.$$

It now follows by an easy diagram chase in the following diagram



that

$$q_m \circ F(f') : X \cdot n \to Q_m$$

coequalizes

$$F(+_n) = !_n \circ a, F(-_n) = !_n \circ b : X \to X \cdot n.$$

So we then get a unique map

$$F(f) : Q_n \to Q_m$$

such that

$$F(f) \circ q_n = q_m \circ F(f'),$$

as indicated in the above diagram.

Now we verify that the choice of the lift f' of $f \circ \mathrm{Coker}(+_n, -_n)$ does not affect the definition of $F(f)$, i.e. that if we define the lift $f^* : [n] \to [m]$ such that $f^*(j) = +$ if $f(\mathrm{Coker}(+_n, -_n))(j) = \#$, then we have

$$q_m \circ F(f') = q_m \circ F(f^*).$$

Since the domain is the coproduct $X \cdot n$, it suffices to show for any $1 \leq j \leq n$ that

$$q_m \circ F(f') \circ \mathrm{in}_j = q_m \circ F(f^*) \circ \mathrm{in}_j.$$

Suppose first that $f(\mathrm{Coker}(+_n, -_n))(j) = \#$, so that $f'(j) = -$ while $f^*(j) = +$. Then we calculate as follows:

$$\begin{aligned} q_m \circ F(f') \circ \mathrm{in}_j &= q_m \circ !_m \circ b \\ &= q_m \circ !_m \circ a \\ &= q_m \circ F(f^*) \circ \mathrm{in}_j, \end{aligned}$$

as desired. And if $f(\mathrm{Coker}(+_n, -_n))(j) = k \in \{1, \ldots, m\}$, then we have

$$F(f') \circ \mathrm{in}_j = \mathrm{in}_k = F(f^*) \circ \mathrm{in}_j,$$

which yields the desired equality. Thus, we have shown that

$$q_m \circ F(f') = q_m \circ F(f^*),$$

so that the definition of $F(f)$ does not depend on the choice of the lift of $f \circ \mathrm{Coker}(+_n, -_n)$.

3. Lastly, let $f : [n] \to [m]_w$ be a map in $\mathcal{B}_w$ from a strictly bipointed set to a pointed set, where we must define

$$F(f) : X \cdot n \to Q_m.$$

Since the domain is a coproduct, it suffices to define

$$F(f) \circ \mathrm{in}_j : X \to Q_m$$

for any $1 \leq j \leq n$.

- If $f(j) = \#$, then we let

$$F(f) \circ \mathrm{in}_j = q_m \circ !_m \circ a = q_m \circ !_m \circ b.$$

- If $f(j) = k \in \{1, \ldots, m\}$, then we let

$$F(f) \circ \mathrm{in}_j = q_m \circ \mathrm{in}_k$$

$$: X \to X \cdot m \to Q_m.$$

This completes the definition of the functor $F : \mathcal{B}_w \to \mathcal{D}$. Now we show that F is actually functorial. First we show that F preserves identity maps. F preserves the identity map of a strictly bipointed set just as in the proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval.

Now let $\mathrm{id} : [n]_w \to [n]_w$ be the identity map of a pointed set, where we show that

$$F(\mathrm{id}) = \mathrm{id} : Q_n \to Q_n.$$

Since the domain is a coequalizer, it suffices to show that

$$F(\mathrm{id}) \circ q_n = q_n.$$

Well, by definition of $F(\mathrm{id})$, we have that

$$F(\mathrm{id}) \circ q_n = q_n \circ F(\mathrm{id}'),$$

where $\mathrm{id}' : [n] \to [n]$ is a lift of $\mathrm{id} \circ \mathrm{Coker}(+_n, -_n) : [n] \to [n]_w$. But it is easily seen that id' can only be the identity on $[n]$, and hence we have

$$\begin{aligned} F(\mathrm{id}) \circ q_n &= q_n \circ F(\mathrm{id}) \\ &= q_n \circ \mathrm{id} \\ &= q_n, \end{aligned}$$

as desired, since we have already seen that F preserves the identity maps of strictly bipointed sets. So F preserves the identity maps of both strictly bipointed and pointed sets, as required.

It has already been shown in the proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval that F preserves the composite of two maps between strictly bipointed sets. Now let $f : [n] \to [m]$ and $g : [m] \to [p]_w$ be maps in $\mathcal{B}_w$, where we show that

$$F(g \circ f) = F(g) \circ F(f) : X \cdot n \to Q_p.$$

Since the domain is a coproduct, it suffices to show that

$$F(g \circ f) \circ \mathrm{in}_j = F(g) \circ F(f) \circ \mathrm{in}_j$$

for any $1 \le j \le n$. We do this by cases as follows:

- First suppose that $g(f(j)) = \#$, because $f(j) = +$ (and $g(+) = \#$, as necessary). Then we have

  $$\begin{aligned} F(g) \circ F(f) \circ \mathrm{in}_j &= F(g) \circ !_m \circ a \\ &= ! \circ a \\ &= q_p \circ !_p \circ a \\ &= F(g \circ f) \circ \mathrm{in}_j, \end{aligned}$$

  as desired. Similarly, if $g(f(j)) = \#$ because $f(j) = -$, then we again have the desired result.

  Now suppose that $g(f(j)) = \#$ because $f(j) = k \in \{1, \ldots, m\}$ and $g(k) = \#$. Then we have

  $$\begin{aligned} F(g) \circ F(f) \circ \mathrm{in}_j &= F(g) \circ \mathrm{in}_k \\ &= q_p \circ !_p \circ a \\ &= F(g \circ f) \circ \mathrm{in}_j, \end{aligned}$$

  as desired.

- Now suppose that $g(f(j)) = q \in \{1, \ldots, p\}$, because $f(j) = k \in \{1, \ldots, m\}$ and $g(k) = q$. Then we have

$$\begin{aligned} F(g) \circ F(f) \circ \mathrm{in}_j &= F(g) \circ \mathrm{in}_k \\ &= q_p \circ \mathrm{in}_q \\ &= F(g \circ f) \circ \mathrm{in}_j, \end{aligned}$$

  as desired.

This completes the proof that $F(g \circ f) = F(g) \circ F(f)$.

Finally, let $f : [n]_w \to [m]_w$ and $g : [m]_w \to [p]_w$ be maps between pointed sets, where we show that

$$F(g \circ f) = F(g) \circ F(f) : Q_n \to Q_p.$$

Since the domain is a coequalizer, it suffices to show that

$$F(g \circ f) \circ q_n = F(g) \circ F(f) \circ q_n.$$

By definition of $F(g \circ f)$, we have that

$$F(g \circ f) \circ q_n = q_p \circ F((g \circ f)'),$$

where $(g \circ f)' : [n] \to [p]$ is a lift of

$$(g \circ f) \circ \mathrm{Coker}(+_n, -_n) : [n] \to [p]_w.$$

Similarly, by definition of $F(f)$ and $F(g)$, we have

$$\begin{aligned} F(g) \circ F(f) \circ q_n &= F(g) \circ q_m \circ F(f') \\ &= q_p \circ F(g') \circ F(f'), \end{aligned}$$

where $g' : [m] \to [p]$ is a lift of

$$g \circ q_m : [m] \to [p]_w,$$

while $f' : [n] \to [m]$ is a lift of

$$f \circ q_n : [n] \to [m]_w.$$

So we will have the desired result if we can show that

$$F((g \circ f)') = F(g') \circ F(f') : X \cdot n \to X \cdot p.$$

But since g' and f' are maps between strictly bipointed sets, this equation follows by the already proven functoriality of F with respect to such maps, once we show that $(g \circ f)' = g' \circ f'$. But this easily follows once we choose the lifts to map integers to the same basepoint, which is possible to do because the choice of lift does not matter, as we have shown.

This completes the proof of the functoriality of $F : \mathcal{B}_w \to \mathcal{D}$. That F satisfies the required commutativity conditions follows in the same way as in the proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval.

Now we show that F preserves finite colimits, by showing that it preserves finite coproducts and coequalizers. First, F preserves the initial object, since we have

$$F([0]) = X \cdot 0 \cong 0.$$

Now we show that F preserves binary coproducts. The fact that F preserves the binary coproduct of two strictly bipointed sets follows from the proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval. Now we show that F preserves the binary coproduct of two pointed sets. So let $n, m \geq 0$, where we show that

$$F([n]_w) + F([m]_w) \cong F([n]_w + [m]_w).$$

Well, because the binary coproduct functor is a left adjoint and hence preserves coequalizers, we have:

$$\begin{aligned} F([n]_w) + F([m]_w) &= Q_n + Q_m \\ &\cong Q_{n+m} \\ &= F([n+m]_w) \\ &= F([n]_w + [m]_w), \end{aligned}$$

as desired.

Now we show that $F([n]_w) + F([m]) \cong F([n]_w + [m])$, i.e. that F preserves the binary coproduct of a pointed set with a strictly bipointed set. First, in any category $\mathcal{E}$ with finite colimits (and any object $X \in \mathcal{E}$) it holds that

$$X \cdot m \cong |\mathrm{Coker}(id_{X \cdot m}, id_{X \cdot m})|.$$

Then since the binary coproduct functor preserves coequalizers, we have

$$\begin{aligned} F([n]_w) + F([m]) &= Q_n + X \cdot m \\ &\cong Q_n + |\mathrm{Coker}(id_{X \cdot m}, id_{X \cdot m})| \\ &\cong Q_{n+m} \\ &= F([n+m]_w) \\ &= F([n]_w + [m]), \end{aligned}$$

as desired.

Now we show that F preserves coequalizers.

- First let $f, g : [n] \to [m]$ be maps between strictly bipointed sets in $\mathcal{B}_w$, with coequalizer $q : [m] \to [p-2]$, where $p \geq 1$ is the number of equivalence classes of $\{1, \ldots, m, -, +\}$ under the equivalence relation R^* generated by the reflexive symmetric relation R (so we are initially supposing that the codomain of q is a strictly bipointed set). Now we want to show that

  $$F(q) : X \cdot m \to X \cdot (p-2)$$

  is the coequalizer in $\mathcal{D}$ of

  $$F(f), F(g) : X \cdot n \to X \cdot m.$$

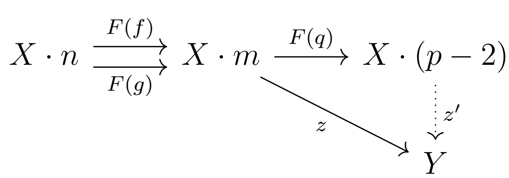
  First, since q coequalizes f and g in $\mathcal{B}_w$, it follows by functoriality of F that $F(q)$ coequalizes $F(f)$ and $F(g)$ in $\mathcal{D}$. Now let $z : X \cdot m \to Y$ be any map in $\mathcal{D}$ such that

  $$z \circ F(f) = z \circ F(g),$$

  where we must construct a unique map

  $$z' : X \cdot (p-2) \to Y$$

  such that $z' \circ F(q) = z$, as in the following diagram:



  Since the domain of z' is a coproduct, it suffices to define

  $$z' \circ \text{in}_k : X \to Y$$

  for any $1 \leq k \leq p-2$. Suppose that k is the number of the equivalence class of k', for some $1 \leq k' \leq m$. Then we let

  $$z' \circ \text{in}_k = z \circ \text{in}_{k'} : X \to Y.$$

  Now we show that z' is well-defined. First we show that if $a, b \in [m]$ are such that aRb, then we have

  $$z \circ \text{in}_a = z \circ \text{in}_b,$$

  where (by definition)

  $$\text{in}_- = !_m \circ b : X \to X \cdot m,$$

  $$\text{in}_+ = !_m \circ a : X \to X \cdot m.$$

If aRb holds because $a = b$, then the result obviously holds. Now suppose that aRb holds because there is some $x \in [n]$ such that $f(x) = a$ and $g(x) = b$ (the other case is analogous). Then we have

$$\begin{aligned} z \circ \mathrm{in}_a &= z \circ F(f) \circ \mathrm{in}_x \\ &= z \circ F(g) \circ \mathrm{in}_x \\ &= z \circ \mathrm{in}_b, \end{aligned}$$

as desired.

To show that z' is well-defined, let $1 \le k \le p-2$ be such that k is the number of the equivalence class of both $k', k'' \in [m]$, where we want to show

$$z \circ \mathrm{in}_{k'} = z \circ \mathrm{in}_{k''}.$$

So we have $k'R^*k''$, and if this holds because $k'Rk''$, then we have just shown that the desired equality holds. Now suppose that $k'R^*k''$ because there are $x_1, \ldots, x_r \in [m]$ (for some $r \ge 1$) such that $k'Rx_1R \ldots Rx_rRk''$. Then we have

$$z \circ \mathrm{in}_{k'} = z \circ \mathrm{in}_{x_1} = \ldots = z \circ \mathrm{in}_{x_r} = z \circ \mathrm{in}_{k''},$$

as desired. Thus, we have shown that $z' : X \cdot (p-2) \to Y$ is well-defined.

Now we must show that $z' \circ F(q) = z$, where it suffices to show

$$z' \circ F(q) \circ \mathrm{in}_{k'} = z \circ \mathrm{in}_{k'} : X \to Y$$

for any $1 \le k' \le m$. First suppose that $q(k') = +$, so that we have

$$z' \circ F(q) \circ \mathrm{in}_{k'} = z' \circ !_{p-2} \circ a = !_Y \circ a.$$

Since $q(k') = +$, this means that $k'R^*+$. First suppose that this relation holds because there is some $x \in [n]$ such that $f(x) = k'$ and $g(x) = +$ (the other case is similar). Then we have

$$\begin{aligned} z \circ \mathrm{in}_{k'} &= z \circ F(f) \circ \mathrm{in}_x \\ &= z \circ F(g) \circ \mathrm{in}_x \\ &= z \circ !_m \circ a \\ &= !_Y \circ a, \end{aligned}$$

as desired. And if $k'R^*+$ holds because there are $x_1, \ldots, x_r \in [m]$ (for some $r \ge 1$) such that $k'Rx_1R \ldots Rx_rR+$, then we have

$$z \circ \mathrm{in}_{k'} = z \circ \mathrm{in}_{x_1} = \ldots = z \circ \mathrm{in}_{x_r} = z \circ !_m \circ a = !_Y \circ a,$$

as desired. Similar reasoning applies in the case where $q(k') = -$. Now suppose that $q(k')$ is not one of the basepoints: then we have

$$z' \circ F(q) \circ \mathrm{in}_{k'} = z' \circ \mathrm{in}_{q(k')} = z \circ \mathrm{in}_{k'},$$

as desired. Thus, we have shown that $z' \circ F(q) = z$.

Finally, we must show that z' is unique such that $z' \circ F(q) = z$. So let $z^* : X \cdot (p-2) \to Y$ be such that $z^* \circ F(q) = z$, where we show $z^* = z'$, and where it suffices to show that

$$z^* \circ \mathrm{in}_k = z' \circ \mathrm{in}_k$$

for any $1 \leq k \leq p-2$. Suppose that k is the number of the equivalence class of k', for some $1 \leq k' \leq m$. Then we have

$$z' \circ \mathrm{in}_k = z \circ \mathrm{in}_{k'} = z^* \circ F(q) \circ \mathrm{in}_{k'} = z^* \circ \mathrm{in}_k,$$

as desired (where the third equality follows because $q(k') = k$).

This completes the proof that $F(q)$ is the coequalizer of $F(f)$ and $F(g)$ in $\mathcal{D}$. A similar proof also works when the codomain of the coequalizer q is a pointed set rather than a strictly bipointed set.

- Now let $f, g : [n] \to [m]_w$ be maps in $\mathcal{B}_w$ with coequalizer $q : [m]_w \to [p-1]_w$, where $p \geq 1$ is as before. We want to show that

$$F(q) : Q_m \to Q_{p-1}$$

is the coequalizer in $\mathcal{D}$ of

$$F(f), F(g) : X \cdot n \to Q_m.$$

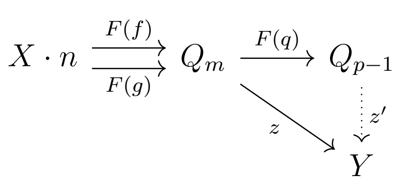
As before, the functoriality of F implies that $F(q)$ coequalizes $F(f)$ and $F(g)$, so now let

$$z : Q_m \to Y$$

be such that $z \circ F(f) = z \circ F(g)$, where we must construct a unique map

$$z' : Q_{p-1} \to Y$$

such that $z' \circ F(q) = z$, as in the following diagram:



Because the domain of z' is a coequalizer, it suffices to define $z^* : X \cdot (p-1) \to Y$ such that

$$z^* \circ\, !_{p-1} \circ a = z^* \circ\, !_{p-1} \circ b.$$

And then it suffices to define

$$z^* \circ \text{in}_k : X \to Y$$

for any $1 \leq k \leq p-1$. So assume that k is the number of the equivalence class of k' (for some $1 \leq k' \leq m$). Then we let

$$z^* \circ \text{in}_k = z \circ q_m \circ \text{in}_{k'}.$$

Now we must show that z^* is well-defined. First we show that for any $a, b \in [m]_w$, if aRb then

$$z \circ q_m \circ \text{in}_a = z \circ q_m \circ \text{in}_b,$$

where

$$\text{in}_\# = !_m \circ a : X \to X \cdot m.$$

If aRb is true because $a = b$, then the desired equality trivially holds. If aRb holds because there is some $x \in [n]$ such that $f(x) = a$ and $g(x) = b$ (the other case is analogous), then we calculate as follows:

$$\begin{aligned} z \circ q_m \circ \text{in}_a &= z \circ F(f) \circ \text{in}_x \\ &= z \circ F(g) \circ \text{in}_x \\ &= z \circ q_m \circ \text{in}_b, \end{aligned}$$

as desired. To show that z^* is well-defined, suppose that k is the number of the equivalence class of both $k, k' \in \{1, \ldots, m\}$, where we wish to show that

$$z \circ q_m \circ \text{in}_{k'} = z \circ q_m \circ \text{in}_{k''}.$$

So we have assumed $k' R^* k''$, and if this is true because $k' R k''$, then we have already shown the result to hold. And if this is true because there are $x_1, \ldots, x_r \in [m]_w$ (for some $r \geq 1$) such that $k' R x_1 R \ldots R x_r R k''$, then we calculate as follows:

$$\begin{aligned} z \circ q_m \circ \text{in}_{k'} &= z \circ q_m \circ \text{in}_{x_1} \\ &\ldots \\ &= z \circ q_m \circ \text{in}_{x_r} \\ &= z \circ q_m \circ \text{in}_{k''}, \end{aligned}$$

as desired. So we have shown that $z^* : X \cdot (p-1) \to Y$ is well-defined. Now we must show that

$$z^* \circ !_{p-1} \circ a = z^* \circ !_{p-1} \circ b,$$

so we calculate as follows:

$$\begin{aligned} z^* \circ !_{p-1} \circ a &= z \circ q_m \circ !_m \circ a \\ &= z \circ q_m \circ !_m \circ b \\ &= z^* \circ !_{p-1} \circ b, \end{aligned}$$

as desired. So we get a unique map

$$z' : Q_{p-1} \to Y$$

such that

$$z' \circ q_{p-1} = z^*.$$

Now we need to show that

$$z' \circ F(q) = z : Q_m \to Y.$$

Since the domain is a coequalizer, it suffices to show that

$$z' \circ F(q) \circ q_m = z \circ q_m : X \cdot m \to Y.$$

And since the domain is now a coproduct, it suffices to show that

$$z' \circ F(q) \circ q_m \circ \mathrm{in}_{k'} = z \circ q_m \circ \mathrm{in}_{k'}$$

$$: X \to Y$$

for any $1 \leq k' \leq m$, i.e. that

$$z' \circ q_{p-1} \circ F(q') \circ \mathrm{in}_{k'} = z \circ q_m \circ \mathrm{in}_{k'},$$

where $q' : [m] \to [p-1]$ is a lift of

$$q \circ \mathrm{Coker}(+_m, -_m) : [m] \to [p-1]_w.$$

So, given $1 \leq k' \leq m$, first suppose that $q(k') = \#$, so that $q'(k') = -$, and also that $k'R^*\#$. First suppose that this latter relation holds because there is some $x \in [n]$ such that $f(x) = k'$ and $g(x) = \#$ (the other case is similar). Then we calculate as follows:

$$\begin{aligned} z' \circ q_{p-1} \circ F(q') \circ \mathrm{in}_{k'} &= z' \circ q_{p-1} \circ !_{p-1} \circ b \\ &= z^* \circ !_{p-1} \circ b \\ &= z \circ q_m \circ !_m \circ b \\ &= z \circ F(g) \circ \mathrm{in}_x \\ &= z \circ F(f) \circ \mathrm{in}_x \\ &= z \circ q_m \circ \mathrm{in}_{k'}, \end{aligned}$$

as desired. Now suppose that $k'R^*\#$ holds because there are $x_1, \ldots, x_r \in [m]_w$

(for some $r \geq 1$) with $k' R x_1 R \dots R x_r R \#$. Then we calculate as follows:

$$\begin{aligned} z \circ q_m \circ \mathrm{in}_{k'} &= z \circ q_m \circ \mathrm{in}_{x_1} \\ &\dots \\ &= z \circ q_m \circ \mathrm{in}_{x_r} \\ &= z \circ q_m \circ \mathrm{in}_{\#} \\ &= z \circ q_m \circ !_m \circ a \\ &= z^* \circ !_{p-1} \circ a \\ &= z' \circ q_{p-1} \circ !_{p-1} \circ a \\ &= z' \circ q_{p-1} \circ !_{p-1} \circ b \\ &= z' \circ q_{p-1} \circ F(q') \circ \mathrm{in}_{k'}, \end{aligned}$$

as desired. Now suppose that $q(k') = k$ for some $1 \leq k \leq p-1$, so that $q'(k') = k$ as well, and we calculate as follows:

$$\begin{aligned} z' \circ q_{p-1} \circ F(q') \circ \mathrm{in}_{k'} &= z' \circ q_{p-1} \circ \mathrm{in}_k \\ &= z^* \circ \mathrm{in}_k \\ &= z \circ q_m \circ \mathrm{in}_{k'}, \end{aligned}$$

as desired. This completes the proof that $z' \circ F(q) = z$.

Finally, we need to show that z' is the unique map such that $z' \circ F(q) = z$. So let

$$\hat{z} : Q_{p-1} \to Y$$

be such that

$$\hat{z} \circ F(q) = z = z' \circ F(q),$$

where we must prove that $z' = \hat{z}$. Since the domain is a coequalizer, it suffices to prove that

$$z' \circ q_{p-1} = \hat{z} \circ q_{p-1},$$

i.e. that

$$z^* = \hat{z} \circ q_{p-1} : X \cdot (p-1) \to Y.$$

Since the domain is a coproduct, it suffices to show that

$$z^* \circ \mathrm{in}_k = \hat{z} \circ q_{p-1} \circ \mathrm{in}_k : X \to Y$$

for any $1 \leq k \leq p-1$. So suppose that k is the number of the equivalence class of $k' \in \{1, \dots, m\}$, where we calculate as follows:

$$\begin{aligned} z^* \circ \mathrm{in}_k &= z \circ q_m \circ \mathrm{in}_{k'} \\ &= \hat{z} \circ F(q) \circ q_m \circ \mathrm{in}_{k'} \\ &= \hat{z} \circ q_{p-1} \circ F(q') \circ \mathrm{in}_{k'} \\ &= \hat{z} \circ q_{p-1} \circ \mathrm{in}_k, \end{aligned}$$

as desired (where the last equality follows because $q'(k') = k$). This completes the proof that $z' = \hat{z}$, and hence that $F(q)$ is the coequalizer of $F(f)$ and $F(g)$.

- A similar proof to the preceding one shows that if $f, g : [n]_w \to [m]_w$ are maps in $\mathcal{B}_w$ with coequalizer $q : [m]_w \to [p-1]_w$ (with $p \geq 1$ as before), then

$$F(q) : Q_m \to Q_{p-1}$$

is the coequalizer in $\mathcal{D}$ of

$$F(f), F(g) : Q_n \to Q_m.$$

This completes the proof that F preserves (finite) coequalizers, and so we have proven that F preserves all finite colimits. This completes the proof that $\mathcal{B}_w$ satisfies part (1) of Proposition 4.12.

□

Now we move on to proving that $\mathcal{B}_w$ satisfies part (2) of Proposition 4.12.

Proposition 4.14. *The category $\mathcal{B}_w$ satisfies part (2) of Proposition 4.12.*

Proof. Let $\mathcal{D}$ be any category with finite colimits and co-interval (X, a, b), with $a, b : X \to 0$. Suppose given finite-colimit-preserving functors $G, H : \mathcal{B}_w \to \mathcal{D}$ with isomorphisms

- $G^* : G([1]) \cong X$ and $G^\# : G([0]) \cong 0$,
- $H^* : H([1]) \cong X$ and $H^\# : H([0]) \cong 0$,

and also suppose given some map $h : G([1]) \to H([1])$ such that both of the following equalities hold:

$$(H^\#)^{-1} \circ G^\# \circ G(\top) = H(\top) \circ h : G([1]) \to H([0]),$$

$$(H^\#)^{-1} \circ G^\# \circ G(\bot) = H(\bot) \circ h : G([1]) \to H([0]).$$

We want to show that there is a unique natural transformation $\theta : G \to H$ such that

$$\theta_{[1]} = h : G([1]) \to H([1]).$$

First, we let

$$\theta_{[0]} = (H^\#)^{-1} \circ G^\# : G([0]) \to H([0])$$

and we let

$$\theta_{[1]} = h : G([1]) \to H([1]).$$

For any strictly bipointed set $[n] = [1] \cdot n$ for $n \geq 2$, we define

$$\theta_{[n]} : G([n]) \to H([n])$$

in exactly the same way as in the proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval. We repeat that definition for explicitness as follows. Since G and H are finite-colimit-preserving and hence finite-coproduct-preserving, we have the following two isomorphisms in $\mathcal{D}$:

$$G^n : G([1] \cdot n) \cong G([1]) \cdot n,$$

$$H^n : H([1] \cdot n) \cong H([1]) \cdot n.$$

Also, we have $h : G([1]) \to H([1])$, and hence we have

$$[\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] : G([1]) \cdot n \to H([1]) \cdot n,$$

where $\mathrm{in}_{H,j} : H([1]) \to H([1]) \cdot n$ for $1 \leq j \leq n$ is an injection map. Finally, we let

$$\theta_{[n]} = (H^n)^{-1} \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,n} \circ h] \circ G^n$$

$$: G([n]) = G([1] \cdot n) \to H([1] \cdot n) = H([n]),$$

as in the following commutative diagram:

$$\begin{array}{ccc} G([n]) & \xrightarrow{\theta_{[n]}} & H([n]) \\ {\scriptstyle G^n}\big\downarrow & & \big\uparrow{\scriptstyle (H^{\#})^{-1}} \\ G([1] \cdot n) & \xrightarrow[{[\mathrm{in}_1 \circ h, \ldots, \mathrm{in}_n \circ h]}]{} & H([1] \cdot n) \end{array}$$

Now we define

$$\theta_{[n]_w} : G([n]_w) \to H([n]_w)$$

for any $n \geq 0$ as follows. First, it was noted in the proof of the preceding Proposition 4.9 that

$$[n]_w = |\mathrm{Coker}(+_n, -_n)|,$$

where $+_n, -_n : [1] \to [n]$ are such that $+_n(1) = +$ and $-_n(1) = -$. Then it is easily seen that we have both

$$+_n = !_n \circ \top : [1] \to [0] \to [n],$$

$$-_n = !_n \circ \bot : [1] \to [0] \to [n].$$

So it follows that

$$[n]_w = |\mathrm{Coker}(!_n \circ \top, !_n \circ \bot)|.$$

Then because G and H preserve finite colimits, we have both

$$G([n]_w) = G(|\mathrm{Coker}(!_n \circ \top, !_n \circ \bot)|) \cong |\mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot))|,$$

$$H([n]_w) = H(|\mathrm{Coker}(!_n \circ \top, !_n \circ \bot)|) \cong |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|.$$

So it suffices to define

$$\theta_{[n]_w} : |\mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot))| \to |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|,$$

and for this, it suffices to define

$$\overline{\theta_{[n]_w}} : G([n]) \to |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|$$

such that

$$\overline{\theta_{[n]_w}} \circ G(!_n \circ \top) = \overline{\theta_{[n]_w}} \circ G(!_n \circ \bot),$$

as in the following commutative diagram:

$$\begin{array}{ccccc} G([1]) & \overset{G(!_n\circ\top)}{\underset{G(!_n\circ\bot)}{\rightrightarrows}} & G([n]) & \xrightarrow{\mathrm{Coker}} & |\mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot))| \\ & & & \searrow^{\overline{\theta_{[n]_w}}} & \Big\downarrow{\scriptstyle \theta_{[n]_w}} \\ & & & & |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))| \end{array}$$

So, we let

$$\overline{\theta_{[n]_w}} = \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \theta_{[n]}$$
$$: G([n]) \to H([n]) \to |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|.$$

Now we must check that

$$\overline{\theta_{[n]_w}} \circ G(!_n \circ \top) = \overline{\theta_{[n]_w}} \circ G(!_n \circ \bot) : G([1]) \to |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|.$$

So we calculate as follows:

$$\begin{aligned} \overline{\theta_{[n]_w}} \circ G(!_n \circ \top) &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \theta_{[n]} \circ G(!_n \circ \top) \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(!_n \circ \top) \circ \theta_{[1]} \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(!_n \circ \bot) \circ \theta_{[1]} \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \theta_{[n]} \circ G(!_n \circ \bot) \\ &= \overline{\theta_{[n]_w}} \circ G(!_n \circ \bot), \end{aligned}$$

as desired, where the second and fourth equalities follow because θ is natural with respect to maps in $\mathcal{B}_w$ between strictly bipointed sets, as shown below. So now we can let

$$\theta_{[n]_w} : |\mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot))| \to |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|$$

be the unique map such that

$$\theta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) = \overline{\theta_{[n]_w}}.$$

This completes the definition of $\theta : G \to H$, and we must now show that θ is natural. The proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval establishes that θ is natural with respect to any map $f : [n] \to [m]$ between strictly bipointed sets.

Now we show that θ is natural with respect to any map $k : [n] \to [m]_w$ from a strictly bipointed set to a pointed set. So we must show that

$$\theta_{[m]_w} \circ G(k) = H(k) \circ \theta[n] : G([n]) \to |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|,$$

as in the following diagram:

$$\begin{array}{ccc} G([n]) & \xrightarrow{\theta_{[n]}} & H([n]) \\ {\scriptstyle G(k)}\downarrow & & \downarrow{\scriptstyle H(k)} \\ G([m]_w) & \xrightarrow[\theta_{[m]_w}]{} & H([m]_w). \end{array}$$

If $n = 0$, then we trivially have

$$\theta_{[m]_w} \circ G(k) = H(k) \circ \theta_{[0]},$$

since $[0]$ is initial in $\mathcal{B}_w$ and G preserves the initial object.

Now suppose that $k : [1] \to [0]_w$, so that

$$k = ! \circ \top = ! \circ \bot : [1] \to [0] \to [0]_w.$$

Then we must show that

$$\theta_{[0]_w} \circ G(! \circ \top) = H(! \circ \top) \circ \theta_{[1]},$$

i.e. that

$$\theta_{[0]_w} \circ G(!) \circ G(\top) = H(!) \circ H(\top) \circ h.$$

Now, $! : [0] \to [0]_w$ is such that $! = \mathrm{Coker}(\top, \bot)$ (with $\top, \bot : [1] \to [0]$), and so we want to show

$$\theta_{[0]_w} \circ G(\mathrm{Coker}(\top, \bot)) \circ G(\top) = H(\mathrm{Coker}(\top, \bot)) \circ H(\top) \circ h.$$

Then since G and H preserve coequalizers, it is equivalent to show that

$$\theta_{[0]_w} \circ \mathrm{Coker}(G(\top), G(\bot)) \circ G(\top) = \mathrm{Coker}(H(\top), H(\bot)) \circ H(\top) \circ h,$$

i.e. (by definition of $\theta_{[0]_w}$) that

$$\mathrm{Coker}(H(\top), H(\bot)) \circ \theta_{[0]} \circ G(\top) = \mathrm{Coker}(H(\top), H(\bot)) \circ H(\top) \circ h.$$

But this holds because (by definition) $\theta_{[0]} = (H^{\#})^{-1} \circ G^{\#}$ and (by assumption on G, H)

$$(H^{\#})^{-1} \circ G^{\#} \circ G(\top) = H(\top) \circ h.$$

Now suppose that $k : [1] \to [n]_w$ (for some $n > 0$), and suppose first that $k(1) = \#$, so that

$$k = \mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ !_n \circ \top : [1] \to [0] \to [n] \to [n]_w.$$

Then we must show that

$$\begin{aligned}\theta_{[n]_w} \circ G(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ !_n \circ \top) &= H(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ !_n \circ \top) \circ \theta_{[1]} \\ &= H(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ !_n \circ \top) \circ h.\end{aligned}$$

By functoriality of G, H and since G, H preserve finite colimits, this is equivalent to showing that

$$\begin{aligned}&\theta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) \circ G(!_n \circ \top) \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(!_n \circ \top) \circ h,\end{aligned}$$

so we calculate as follows:

$$\begin{aligned}\theta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) \circ G(!_n \circ \top) &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \theta_{[n]} \circ G(!_n \circ \top) \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(!_n \circ \top) \circ \theta_{[1]} \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(!_n \circ \top) \circ h,\end{aligned}$$

as desired. Thus, in this case on k, we have shown that $\theta_{[n]_w} \circ G(k) = H(k) \circ \theta_{[1]}$.

Still supposing that $k : [1] \to [n]_w$, assume now that $k(1) = j \in \{1, \ldots, n\}$, so that

$$k = \mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ k' : [1] \to [n] \to [n]_w,$$

where $k' : [1] \to [n]$ is such that $k'(1) = j$. So we want to show that

$$\begin{aligned}\theta_{[n]_w} \circ G(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ k') &= H(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ k') \circ \theta_{[1]} \\ &= H(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot) \circ k') \circ h.\end{aligned}$$

By functoriality of G, H and since G, H preserve colimits, this is equivalent to showing that

$$\theta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) \circ G(k') = \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(k') \circ h.$$

So we calculate as follows:

$$\begin{aligned}\theta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) \circ G(k') &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \theta_{[n]} \circ G(k') \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(k') \circ \theta_{[1]} \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ H(k') \circ h,\end{aligned}$$

as desired. This is the second and final case on $k : [1] \to [n]_w$, and in both cases we have

$$\theta_{[n]_w} \circ G(k) = H(k) \circ \theta_{[1]},$$

as required.

Now suppose that $k : [m] \to [n]_w$ for some $m > 1, n > 0$. Then since $[m] = [1]\cdot m$, it follows that $k = [k \circ \mathrm{in}_1, \ldots, k \circ \mathrm{in}_m]$, where

$$k \circ \mathrm{in}_j : [1] \to [m] \to [n]_w$$

for any $1 \leq j \leq m$. We have already shown that θ is natural with respect to any map $k' : [1] \to [n]_w$, and so in particular we have for any $1 \leq j \leq m$

$$\theta_{[n]_w} \circ G(k \circ \mathrm{in}_j) = H(k \circ \mathrm{in}_j) \circ h.$$

Now we want to show that

$$\theta_{[n]_w} \circ G([k \circ \mathrm{in}_1, \ldots, k \circ \mathrm{in}_m]) = H([k \circ \mathrm{in}_1, \ldots, k \circ \mathrm{in}_m]) \circ \theta_{[m]}.$$

Since G, H preserve finite coproducts, this is equivalent to showing that

$$\theta_{[n]_w} \circ [G(k \circ \mathrm{in}_1), \ldots, G(k \circ \mathrm{in}_m)] \circ G^m = [H(k \circ \mathrm{in}_1), \ldots, H(k \circ \mathrm{in}_m)] \circ H^m \circ \theta_{[m]},$$

so we calculate as follows:

$$\begin{aligned}
\theta_{[n]_w} \circ [G(k \circ \mathrm{in}_1), \ldots G(k \circ \mathrm{in}_m)] \circ G^m &= [\theta_{[n]_w} \circ G(k \circ \mathrm{in}_1), \ldots, \theta_{[n]_w} \circ G(k \circ \mathrm{in}_m)] \circ G^m \\
&= [H(k \circ \mathrm{in}_1) \circ h, \ldots, H(k \circ \mathrm{in}_m) \circ h] \circ G^m \\
&= [[H(k \circ \mathrm{in}_1), \ldots, H(k \circ \mathrm{in}_m)] \circ \mathrm{in}_{H,1} \circ h, \ldots, \\
&\quad [H(k \circ \mathrm{in}_1), \ldots, H(k \circ \mathrm{in}_m)] \circ \mathrm{in}_{H,m} \circ h] \circ G^m \\
&= [H(k \circ \mathrm{in}_1), \ldots, H(k \circ \mathrm{in}_m)] \circ [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \\
&\quad \circ G^m \\
&= [H(k \circ \mathrm{in}_1), \ldots, H(k \circ \mathrm{in}_m)] \circ H^m \circ (H^m)^{-1} \circ \\
&\quad [\mathrm{in}_{H,1} \circ h, \ldots, \mathrm{in}_{H,m} \circ h] \circ G^m \\
&= [H(k \circ \mathrm{in}_1), \ldots, H(k \circ \mathrm{in}_m)] \circ H^m \circ \theta_{[m]},
\end{aligned}$$

as desired. So we have thus shown that

$$\theta_{[n]_w} \circ G(k) = H(k) \circ \theta_{[m]},$$

which completes the proof that θ is natural with respect to any map from a strictly bipointed set to a pointed set.

Lastly, we show that θ is natural with respect to any map $g : [n]_w \to [m]_w$ between pointed sets. So we must show that

$$\theta_{[m]_w} \circ G(g) = H(g) \circ \theta_{[n]_w}$$

$$: |\mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot))| \to |\mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot))|,$$

as in the following diagram:

$$\begin{array}{ccc} G([n]_w) & \xrightarrow{\theta_{[n]_w}} & H([n]_w) \\ {\scriptstyle G(g)}\downarrow & & \downarrow{\scriptstyle H(g)} \\ G([m]_w) & \xrightarrow[\theta_{[m]_w}]{} & H([m]_w) \end{array}$$

Since $\mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot))$ is epi, it suffices to show that

$$\begin{aligned} \theta_{[m]_w} \circ G(g) \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) &= H(g) \circ \theta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) \\ &= H(g) \circ \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \theta_{[n]}. \end{aligned}$$

Since G, H preserve finite colimits, this is equivalent to showing that

$$\theta_{[m]_w} \circ G(g) \circ G(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot)) = H(g) \circ H(\mathrm{Coker}(!_n \circ \top, !_n \circ \bot)) \circ \theta_{[n]},$$

i.e. that

$$\theta_{[m]_w} \circ G(g \circ \mathrm{Coker}(!_n \circ \top, !_n \circ \bot)) = H(g \circ \mathrm{Coker}(!_n \circ \top, !_n \circ \bot)) \circ \theta_{[n]}.$$

But this follows because θ is natural with respect to any map from a strictly bipointed set to a pointed set, as we have already demonstrated. So θ is natural with respect to any map between pointed sets, and this completes the proof of the naturality of $\theta : G \to H$.

To complete the proof that $\mathcal{B}_w$ satisfies part (2) of Proposition 4.12, we need to show that $\theta : G \to H$ as just defined is the unique natural transformation from G to H such that

$$\theta_{[1]} = h : G([1]) \to H([1]).$$

So let $\eta : G \to H$ be a natural transformation with $\eta_{[1]} = h$, where we show that $\eta = \theta$. The proof that $\mathcal{B}$ is the free finite-coproduct category on a co-interval already establishes that $\eta_{[n]} = \theta_{[n]}$ for any strictly bipointed set $[n]$.

Now we want to show that for any pointed set $[n]_w$, we have $\eta_{[n]_w} = \theta_{[n]_w}$. We have defined $\theta_{[n]_w}$ as the unique map such that

$$\begin{aligned} \theta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \theta_{[n]} \\ &= \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \eta_{[n]}, \end{aligned}$$

and so it suffices to show that

$$\eta_{[n]_w} \circ \mathrm{Coker}(G(!_n \circ \top), G(!_n \circ \bot)) = \mathrm{Coker}(H(!_n \circ \top), H(!_n \circ \bot)) \circ \eta_{[n]}.$$

But this follows by the assumption that G, H preserve finite colimits and the assumed naturality of η. Thus, we have $\eta_{[n]_w} = \theta_{[n]_w}$ for any pointed set $[n]_w$, which completes the proof that $\eta = \theta$. Thus, $\theta : G \to H$ is the unique natural transformation such that $\theta_{[1]} = h$. This completes the proof that $\mathcal{B}_w$ satisfies part (2) of Proposition 4.12.

□

Combining Propositions 4.13 and 4.14, we now obtain the following

Theorem 4.15. *The category $\mathcal{B}_w$ is the free finite-colimit category on a co-interval.*

And by duality, we have

Corollary 4.16. *The category $\mathbb{C}_w \doteq \mathcal{B}_w^{op}$ is the free finite-limit category on an interval.*

In this chapter, we have shown that the category $\mathcal{B}$ of finite strictly bipointed sets has the classifying property of being the free finite-coproduct category on a co-interval, and that the dual category $\mathbb{C}$ of Cartesian cubes has the dual classifying property of being the free finite-product category on an interval. We then introduced the category $\mathcal{B}_w$ of finite weakly bipointed sets, and proved that it has the classifying property of being the free finite-colimit category on a co-interval, which entailed that the opposite category $\mathbb{C}_w \doteq \mathcal{B}_w^{op}$ has the dual classifying property of being the free finite-limit category on an interval.

Chapter 5

Conclusion

In this thesis we have introduced the Cartesian category $\mathbb{C}$ of higher-dimensional cubes, which is an extension of the classical cube category. We then proved that this category is dual (by isomorphism) to the category $\mathcal{B}$ of finite strictly bipointed sets,

$$\mathbb{C} \cong \mathcal{B}^{op}.$$

We next showed that $\mathbb{C}$ has the classifying property of being the free finite-product category on an interval, which roughly means that any interval in any finite-product category is the image of the universal interval of $\mathbb{C}$ in an essentially unique way. The proof that $\mathbb{C}$ has this classifying property proceeded by proving that $\mathcal{B}$ has the dual classifying property of being the free finite-coproduct category on a co-interval, and then appealing to the established duality between $\mathbb{C}$ and $\mathcal{B}$. Lastly, we extended the category $\mathcal{B}$ of finite *strictly* bipointed sets to the category $\mathcal{B}_w$ of finite *weakly* bipointed sets, and showed that this category has the classifying property of being the free finite-colimit category on a co-interval. This implied that the dual category $\mathbb{C}_w \doteq \mathcal{B}_w^{op}$ has the classifying property of being the free finite-limit category on an interval.

There are a number of directions in which one could further the research presented in this thesis. Some of these are listed as follows.

- First, the category $\mathbb{C}_w$ has been defined merely as the opposite of the category $\mathcal{B}_w$ of finite weakly bipointed sets. But just as the Cartesian category of cubes $\mathbb{C}$ is the opposite of the category $\mathcal{B}$ of finite strictly bipointed sets (as proven in this thesis), one may regard the category $\mathbb{C}_w$ in a more direct manner as a category of cubes with some additional structure beyond that of $\mathbb{C}$. Specifically, since $\mathcal{B}_w$ has finite colimits, $\mathbb{C}_w$ has finite limits, and in particular (finite) equalizers of cube maps $I^n \rightrightarrows I^m$, beyond just the finite products of $\mathbb{C}$. It would be interesting to further investigate the explicit structure of this category $\mathbb{C}_w$.

- Next, the topos of presheaves $\mathrm{Sets}^{\mathbb{C}_w^{op}}$ is a classifying topos (see [8]) for (weakly) bipointed objects. Using the results of this thesis, one can then also show that $\mathrm{Sets}^{\mathbb{C}^{op}}$ is the classifying topos for *strictly* bipointed objects.

- Since it is easily seen that $\mathbb{C}$ is a (non-full) subcategory of the category Pos of posets and monotone maps, and since Pos in turn is a (full) subcategory of the category Cat of all small categories and functors, one could further explore the relationships between $\mathbb{C}$, Pos, and Cat:

$$\mathbb{C} \hookrightarrow \text{Pos} \hookrightarrow \text{Cat}.$$

- One may investigate the implications of $\mathbb{C}$ being the free finite-product category on an interval. For instance, consider the category Top of topological spaces and continuous functions. As a finite-product category, Top has an interval, namely the closed unit interval $[0,1]$ with the obvious points from the terminal object $\{*\}$ (picking out the endpoints $0,1$). Then since $\mathbb{C}$ is the free finite-product category on an interval (with distinguished interval $(I,\top,\bot)$), there exists a finite-product-preserving functor $F : \mathbb{C} \to \text{Top}$ such that $F(I) \cong [0,1]$, which entails that $F(I^n) \cong [0,1]^n$ (since F preserves finite products) for any $n \geq 0$. If we now consider the Yoneda embedding $y : \mathbb{C} \to \text{Sets}^{\mathbb{C}^{op}}$, then since $\text{Sets}^{\mathbb{C}^{op}}$ is the free co-completion of $\mathbb{C}$ and Top is cocomplete, the functor $F : \mathbb{C} \to \text{Top}$ induces a unique finite-colimit-preserving functor $F_! : \text{Sets}^{\mathbb{C}^{op}} \to \text{Top}$, as in the following commutative diagram:

$$\begin{array}{ccc} \mathbb{C} & \overset{y}{\hookrightarrow} & \text{Sets}^{\mathbb{C}^{op}} \\ & \underset{F}{\searrow} & \vdots\, F_! \\ & & \text{Top} \end{array}$$

  The action of $F_!$ on a presheaf $X : \mathbb{C}^{op} \to \text{Sets}$ can be seen as follows. First, we know that X is a colimit of representable presheaves:

$$X \cong \operatorname*{colim}_n y(I^n).$$

  Then (remembering that $F_!$ preserves colimits) we may calculate as follows:

$$\begin{aligned} F_!(X) &\cong F_!(\operatorname*{colim}_n y(I^n)) \\ &\cong \operatorname*{colim}_n F_!(y(I^n)) \\ &\cong \operatorname*{colim}_n F(I^n) \\ &\cong \operatorname*{colim}_n [0,1]^n. \end{aligned}$$

  This defines the action of $F_!$ on a presheaf $X : \mathbb{C}^{op} \to \text{Sets}$, where $F_!(X)$ is called the geometric realization of the cubical set X, which is a construction important in homotopy theory.

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