Free algebras of topologically enriched multi-sorted equational theories

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Introduction

- Classical *multi-sorted equational theories* and their *free algebras* have been fundamental in mathematics and computer science: e.g. in studying algebraic specification [Mit96], computational effects [PP04], and algebraic databases and data integration [SSVW17, SW17].
- Classical multi-sorted equational theories are **Set-enriched**: their algebras are multi-sorted sets equipped with finitary operations that must satisfy certain equations. There is a well-known explicit and constructive description of their free algebras.

Introduction

- In this talk, given a symmetric monoidal category $\mathscr{V}$ that is **topological over Set**, I will define a notion of $\mathscr{V}$**-enriched multi-sorted equational theory**: the algebras will be multi-sorted **objects of** $\mathscr{V}$ equipped with $\mathscr{V}$**-parameterized** finitary operations that must satisfy certain equations.

- Every $\mathscr{V}$-enriched multi-sorted equational theory $\mathcal{T}$ has an underlying classical multi-sorted equational theory $|\mathcal{T}|$, and (because $\mathscr{V}$ is topological over **Set**) free $\mathcal{T}$-algebras can be explicitly described as suitable "liftings" of free $|\mathcal{T}|$-algebras. I will provide some examples of $\mathscr{V}$-enriched multi-sorted equational theories, and explain their connection to $\mathscr{V}$**-enriched algebraic theories and monads for a subcategory of arities** (when $\mathscr{V}$ is symmetric monoidal closed).

Review of classical multi-sorted equational theories

- Fix a set $\mathcal{S}$ of **sorts**. A **(classical)** $\mathcal{S}$**-sorted signature** is a set of operation symbols Σ equipped with an assignment to each $\sigma \in \Sigma$ of a finite tuple $(S_1, \ldots, S_n)$ of **input sorts** and an **output sort** S:

$$\sigma : S_1 \times \ldots \times S_n \to S.$$

- Given a context $\vec{v} \equiv v_1 : T_1, \ldots, v_m : T_m$ of $\mathcal{S}$-sorted variables, for each sort $S \in \mathcal{S}$ we can define the set $\mathbf{Term}\,(\Sigma; \vec{v})_S$ of Σ-**terms** $[\vec{v} \vdash t : S]$ **of sort** S **in context** $\vec{v}$ in a standard way.

Review of classical multi-sorted equational theories

- A Σ-**equation in context** $[\vec{v} \vdash s \doteq t : S]$ consists of a context $\vec{v}$ and two Σ-terms s, t of the same sort S in context $\vec{v}$. A **(classical)** $\mathcal{S}$**-sorted equational theory** is a pair $\mathcal{T} = (\Sigma, \mathcal{E})$ consisting of a classical $\mathcal{S}$-sorted signature Σ and a set $\mathcal{E}$ of Σ-equations in context.

- A Σ-**algebra** $\mathbb{A}$ is an $\mathcal{S}$-sorted **carrier set** $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$ (i.e. an object of $\mathbf{Set}^{\mathcal{S}}$) equipped with, for each $\sigma \in \Sigma$, a function

$$\sigma^{\mathbb{A}} : A_{S_1} \times \ldots \times A_{S_n} \to A_S.$$

Given a context $\vec{v} \equiv v_1 : T_1, \ldots, v_m : T_m$, each Σ-term $[\vec{v} \vdash t : S]$ of sort S in context $\vec{v}$ induces an interpretation function

$$[\vec{v} \vdash t : S]^{\mathbb{A}} : A_{T_1} \times \ldots \times A_{T_m} \to A_S.$$

Review of classical multi-sorted equational theories

- A Σ-algebra $\mathbb{A}$ **satisfies** a Σ-equation in context $[\vec{v} \vdash s \doteq t : S]$ if $[\vec{v} \vdash s : S]^{\mathbb{A}} = [\vec{v} \vdash t : S]^{\mathbb{A}}$. Given a classical $\mathcal{S}$-sorted equational theory $\mathcal{T} = (\Sigma, \mathcal{E})$, a Σ-algebra $\mathbb{A}$ is a $\mathcal{T}$-**algebra** or $\mathcal{T}$-**model** if $\mathbb{A}$ satisfies each Σ-equation in $\mathcal{E}$.
- We obtain a category Σ-**Alg** of Σ-algebras and their morphisms and a forgetful functor $U^{\Sigma} : \Sigma\text{-}\mathbf{Alg} \to \mathbf{Set}^{\mathcal{S}}$. Given $\mathcal{T} = (\Sigma, \mathcal{E})$, we have the full subcategory $\mathcal{T}\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$ and the restricted forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to \mathbf{Set}^{\mathcal{S}}$.

Free algebras of classical multi-sorted equational theories

- Given a classical $\mathcal{S}$-sorted equational theory $\mathcal{T} = (\Sigma, \mathcal{E})$, the forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to \mathbf{Set}^{\mathcal{S}}$ has a left adjoint $F^{\mathcal{T}} : \mathbf{Set}^{\mathcal{S}} \to \mathcal{T}\text{-}\mathbf{Alg}$ with the following well-known explicit, constructive description.

- First, we have the following description of the left adjoint $F^{\Sigma} : \mathbf{Set}^{\mathcal{S}} \to \Sigma\text{-}\mathbf{Alg}$ of $U^{\Sigma} : \Sigma\text{-}\mathbf{Alg} \to \mathbf{Set}^{\mathcal{S}}$. Given an $\mathcal{S}$-sorted set $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$, one considers the classical $\mathcal{S}$-sorted signature $\Sigma_{\mathcal{A}}$ obtained from Σ by adjoining, for each sort $S \in \mathcal{S}$ and each $a \in A_S$, a new constant symbol $c_a : S$. The $\mathcal{S}$-sorted set $\mathbf{Term}\,(\Sigma_{\mathcal{A}}; \varnothing)$ of **ground** $\Sigma_{\mathcal{A}}$**-terms** carries the structure of the free Σ-algebra $F^{\Sigma}\mathcal{A}$, with $\sigma^{F^{\Sigma}\mathcal{A}}$ given by $(t_1, \ldots, t_n) \mapsto \sigma(t_1, \ldots, t_n)$ for each $\sigma \in \Sigma$.

- To construct the free $\mathcal{T}$-algebra $F^{\mathcal{T}}\mathcal{A}$ on $\mathcal{A}$, one considers the smallest Σ-**congruence** $\sim^{\mathcal{E}}$ **on** $F^{\Sigma}\mathcal{A}$ **generated by** $\mathcal{E}$. The **quotient** Σ-**algebra** $F^{\Sigma}\mathcal{A}/\!\sim^{\mathcal{E}}$ is then the free $\mathcal{T}$-algebra on $\mathcal{A}$.

Enrichment of classical multi-sorted equational theories

- To develop an enriched notion of multi-sorted equational theory for which free algebras can (still) be explicitly and constructively described, we consider a symmetric monoidal category $\mathscr{V} = (\mathscr{V}, \otimes, I)$ such that the representable functor $| - | := \mathscr{V}(I, -) : \mathscr{V} \to \mathbf{Set}$ is strict monoidal and **topological**.
- Examples (other than **Set**) include:
  - Various categories of topological spaces, and the category of measurable spaces;
  - The categories of models of **relational Horn theories without equality**, including the categories of preordered sets and (extended) pseudo-metric spaces;
  - The categories of **quasispaces** (a.k.a. **concrete sheaves**) on **concrete sites** [BH11, Dub79, MMS22], including diffeological spaces, bornological sets, simplicial complexes, pseudotopological spaces, and convergence spaces.
  - Many of the categories studied in **monoidal topology** [HST14].

$\mathscr{V}$-enriched multi-sorted signatures

- Fix a set $\mathcal{S}$ of **sorts**. A $\mathscr{V}$**-enriched** $\mathcal{S}$**-sorted signature** is a set of operation symbols Σ equipped with an assignment to each $\sigma \in \Sigma$ of a finite tuple $(S_1, \ldots, S_n)$ of **input sorts**, an **output sort** S, and a **parameter object** $P \in \mathbf{ob}(\mathscr{V})$; we say that σ has **type** $((S_1, \ldots, S_n), S, P)$.

- A Σ**-algebra** $\mathbb{A}$ is an $\mathcal{S}$-sorted **carrier object** $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$ of $\mathscr{V}$, i.e. an object of $\mathscr{V}^{\mathcal{S}}$, equipped with, for each $\sigma \in \Sigma$ as above, a $\mathscr{V}$*-morphism*

$$\sigma^{\mathbb{A}} : P \otimes (A_{S_1} \times \ldots \times A_{S_n}) \to A_S.$$

  We obtain a category Σ-**Alg** of Σ-algebras and their morphisms, and a forgetful functor $U^{\Sigma} : \Sigma\text{-}\mathbf{Alg} \to \mathscr{V}^{\mathcal{S}}$.

$\mathscr{V}$-enriched multi-sorted equational theories

- Every $\mathscr{V}$-enriched $\mathcal{S}$-sorted signature Σ has an **underlying classical $\mathcal{S}$-sorted signature** $|\Sigma|$. For each $\sigma \in \Sigma$ of type $\left(\overline{S}, S, P\right)$ and each $p \in |P|$, $|\Sigma|$ has an operation symbol σ_p with input sorts $\overline{S}$ and output sort S. We can then consider $|\Sigma|$-equations in context. Moreover, every Σ-algebra $\mathbb{A}$ has an underlying $|\Sigma|$-algebra $|\mathbb{A}|$, and we obtain a functor $| - |^{\Sigma} : \Sigma\text{-}\mathbf{Alg} \to |\Sigma|\text{-}\mathbf{Alg}$.

- A **$\mathscr{V}$-enriched $\mathcal{S}$-sorted equational theory** is a pair $\mathcal{T} = (\Sigma, \mathcal{E})$ consisting of a $\mathscr{V}$-enriched $\mathcal{S}$-sorted signature Σ and a set $\mathcal{E}$ of $|\Sigma|$-equations in context, so that $|\mathcal{T}| := (|\Sigma|, \mathcal{E})$ is a classical $\mathcal{S}$-sorted equational theory. A **$\mathcal{T}$-algebra** is a Σ-algebra $\mathbb{A}$ whose underlying $|\Sigma|$-algebra $|\mathbb{A}|$ is a $|\mathcal{T}|$-algebra (i.e. satisfies every $|\Sigma|$-equation in $\mathcal{E}$). We have the full subcategory $\mathcal{T}\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$ and the restricted forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to \mathscr{V}^{\mathcal{S}}$.

Examples of $\mathscr{V}$-enriched multi-sorted equational theories

- Every classical multi-sorted equational theory $\mathcal{T}$ determines a $\mathscr{V}$-enriched multi-sorted equational theory $\mathcal{T}^*$ for which a $\mathcal{T}^*$-algebra can be described as a $\mathcal{T}$-algebra in $\mathscr{V}$. For example:
  - There are $\mathscr{V}$-enriched **single-sorted** equational theories whose algebras are internal semigroups/monoids/groups in $\mathscr{V}$, commutative ring objects in $\mathscr{V}$, ...
  - When $(\mathscr{V}, \otimes, I)$ is cartesian, there is a $\mathscr{V}$-enriched $\mathbb{N}$-sorted equational theory whose algebras are (symmetric) $\mathscr{V}$-based operads.
  - When $(\mathscr{V}, \otimes, I)$ is cartesian, for each fixed set $\mathcal{O}$ there is a $\mathscr{V}$-enriched $(\mathcal{O} \times \mathcal{O})$-sorted equational theory whose algebras are $\mathscr{V}$-categories with object set $\mathcal{O}$.
  - For each small category $\mathscr{A}$, there is a $\mathscr{V}$-enriched $\mathbf{ob}(\mathscr{A})$-sorted equational theory whose algebras are functors $\mathscr{A} \to \mathscr{V}$.

Examples of $\mathscr{V}$-enriched multi-sorted equational theories

- Let $\mathscr{V}$ be a suitable category of topological spaces. There is a $\mathscr{V}$-enriched single-sorted equational theory for which an algebra may be described as a (strict/coherent) H-**space**, i.e. an internal 'monoid' in $\mathscr{V}$ whose product is only associative and unital up to specified homotopies. More generally, given any classical $\mathcal{S}$-sorted equational theory $\mathcal{T}$, there is a $\mathscr{V}$-enriched $\mathcal{S}$-sorted equational theory $\mathcal{T}_h$ (the **homotopy weakening of** $\mathcal{T}$) whose algebras may be described as the '$\mathcal{T}$-algebras' in $\mathscr{V}$ that only satisfy the equations of $\mathcal{T}$ up to specified homotopies.

- Suppose that $\mathscr{V}$ is symmetric monoidal *closed*, and let $\mathscr{A}$ be a small $\mathscr{V}$-category. There is a $\mathscr{V}$-enriched $\mathbf{ob}(\mathscr{A})$-sorted equational theory whose algebras are *$\mathscr{V}$-functors* $\mathscr{A} \to \mathscr{V}$.

- Given a relational Horn theory $\mathbb{T}$ without equality, a certain subclass of the **relational algebraic theories** of [FMS21] can be described as $\mathbb{T}$-**Mod**-enriched single-sorted equational theories.

Free algebras of $\mathscr{V}$-enriched multi-sorted equational theories

Given a $\mathscr{V}$-enriched $\mathcal{S}$-sorted equational theory $\mathcal{T} = (\Sigma, \mathcal{E})$ and its underlying classical $\mathcal{S}$-sorted equational theory $|\mathcal{T}| = (|\Sigma|, \mathcal{E})$, the functor $|-|^{\Sigma} : \Sigma\text{-}\mathbf{Alg} \to |\Sigma|\text{-}\mathbf{Alg}$ restricts to a functor $|-|^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to |\mathcal{T}|\text{-}\mathbf{Alg}$.

Theorem ([Par23])

The forgetful functor $U^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to \mathscr{V}^{\mathcal{S}}$ *has a left adjoint* $F^{\mathcal{T}} : \mathscr{V}^{\mathcal{S}} \to \mathcal{T}\text{-}\mathbf{Alg}$*, and the resulting adjunction* $F^{\mathcal{T}} \dashv U^{\mathcal{T}} : \mathcal{T}\text{-}\mathbf{Alg} \to \mathscr{V}^{\mathcal{S}}$ *is a (strict) lifting of the adjunction* $F^{|\mathcal{T}|} \dashv U^{|\mathcal{T}|} : |\mathcal{T}|\text{-}\mathbf{Alg} \to \mathbf{Set}^{\mathcal{S}}$*. In particular, the following diagram strictly commutes:*

$$\begin{array}{ccc} \mathscr{V}^{\mathcal{S}} & \xrightarrow{F^{\mathcal{T}}} & \mathcal{T}\text{-}\mathbf{Alg} \\ {\scriptstyle |-|^{\mathcal{S}}}\big\downarrow & & \big\downarrow{\scriptstyle |-|^{\mathcal{T}}} \\ \mathbf{Set}^{\mathcal{S}} & \xrightarrow{F^{|\mathcal{T}|}} & |\mathcal{T}|\text{-}\mathbf{Alg}. \end{array}$$

Free algebras of $\mathscr{V}$-enriched multi-sorted equational theories

- Thus, for each object $\mathcal{A} = (A_S)_{S \in \mathcal{S}}$ of $\mathscr{V}^{\mathcal{S}}$, the free $\mathcal{T}$-algebra $F^{\mathcal{T}}\mathcal{A}$ on $\mathcal{A}$ lies over the free $|\mathcal{T}|$-algebra $F^{|\mathcal{T}|}\left(|\mathcal{A}|^{\mathcal{S}}\right)$ on the underlying $\mathcal{S}$-sorted set $|\mathcal{A}|^{\mathcal{S}} = (|A_S|)_{S \in \mathcal{S}}$ in $\mathbf{Set}^{\mathcal{S}}$. The free $\mathcal{T}$-algebra $F^{\mathcal{T}}\mathcal{A}$ is completely determined by its carrier object $U^{\mathcal{T}}F^{\mathcal{T}}\mathcal{A}$ of $\mathscr{V}^{\mathcal{S}}$, which equips the carrier object $U^{|\mathcal{T}|}\left(F^{|\mathcal{T}|}\left(|\mathcal{A}|^{\mathcal{S}}\right)\right)$ of $\mathbf{Set}^{\mathcal{S}}$ with an appropriate *$\mathscr{V}$-structure* (e.g. an appropriate topology).

- Using the above Theorem, we can provide an explicit description of this $\mathscr{V}$-structure, and hence of the free $\mathcal{T}$-algebra $F^{\mathcal{T}}\mathcal{A}$ on $\mathcal{A}$. When $\mathscr{V}$ is *cartesian closed*, this explicit description becomes even more constructive and inductive. (See upcoming preprint for details!)

- When $\mathcal{T}$ is single-sorted (and each parameter object is trivial), we recover previously established descriptions of free $\mathcal{T}$-algebras for certain specific $\mathscr{V}$, including [Por88, Por91, Bat10].

The connection with $\mathscr{V}$-enriched algebraic theories and monads

- Given an arbitrary symmetric monoidal closed category $\mathscr{V}$ and a **subcategory of arities** $\mathscr{J} \hookrightarrow \mathscr{C}$ in a $\mathscr{V}$-category $\mathscr{C}$, there is an established theory of **enriched algebraic** $\mathscr{J}$**-theories** and $\mathscr{J}$**-ary/**$\mathscr{J}$**-nervous** $\mathscr{V}$**-monads** on $\mathscr{C}$: see [LW16, LWP22, LWP23a, LWP23b].

- In the current setting, we now suppose that $\mathscr{V}$ is symmetric monoidal *closed*. With $\mathscr{C} = \mathscr{V}^{\mathcal{S}}$, there is a subcategory of arities $\mathscr{J} = \mathbb{N}_{\mathcal{S}} \hookrightarrow \mathscr{V}^{\mathcal{S}}$ whose objects are $\mathcal{J} = (n_S \cdot I)_{S \in \mathcal{S}}$ with all but finitely many $n_S = 0$ $(S \in \mathcal{S})$.

Theorem ([Par23])

There is a correspondence between $\mathscr{V}$-enriched $\mathcal{S}$-sorted equational theories, $\mathbb{N}_{\mathcal{S}}$-theories, and $\mathbb{N}_{\mathcal{S}}$-nervous $\mathscr{V}$-monads on $\mathscr{V}^{\mathcal{S}}$.

In conclusion

- Classical (**Set**-enriched) multi-sorted equational theories and their free algebras have played important roles in mathematics and computer science. Given a symmetric monoidal category $\mathscr{V}$ that is topological over **Set**, we have defined a notion of $\mathscr{V}$-enriched multi-sorted equational theory that recovers the classical notion when $\mathscr{V} = \mathbf{Set}$.

- Because $\mathscr{V}$ is topological over **Set**, free algebras for $\mathscr{V}$-enriched multi-sorted equational theories can be explicitly obtained as suitable "liftings" of free algebras of their underlying classical counterparts. When $\mathscr{V}$ is symmetric monoidal closed, $\mathscr{V}$-enriched multi-sorted equational theories correspond to $\mathscr{V}$-enriched algebraic theories and monads for a certain subcategory of arities.

Thank you!

(Preprint will be on arXiv in the next week or two!)

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