Isotropy Groups of Algebraic Theories

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Overview

In this talk, I will show how we can use categorical universal algebra to define a notion of *inner automorphism* for algebraic theories other than just monoids, groups, rings, etc.

I will first motivate this work by describing a result of George Bergman, then I will explain the general theory, and lastly I will provide some examples.

Motivation

Let G be any group, and let *Group* be the category of groups and group homomorphisms. Then there is a forgetful functor

$$U_G : G \downarrow Group \to Group,$$

which sends any object $h : G \to H$ of $G \downarrow Group$ to the group H.

An automorphism of this functor U_G (i.e. an invertible natural transformation $U_G \Rightarrow U_G$) then consists of the following data (by definition):

1. A group automorphism
$$\alpha : G \xrightarrow{\sim} G.$$
2. $\forall$ morphism $f : G \to H$ in *Group*, a group automorphism
$$\alpha_f : H \xrightarrow{\sim} H$$
such that $\alpha_{id_G} = \alpha$ and for any commuting triangle

Motivation

$$
\begin{array}{ccc}
G & \xrightarrow{f_1} & H_1 \\
 & \searrow^{f_2} & \downarrow h \\
 & & H_2
\end{array}
$$

in *Group*, the following diagram commutes:

$$
\begin{array}{ccc}
H_1 & \xrightarrow{\alpha_{f_1}} & H_1 \\
h \downarrow & & \downarrow h \\
H_2 & \xrightarrow[\alpha_{f_2}]{} & h_2
\end{array}
$$

An automorphism of $U_G : G \downarrow \mathit{Group} \to \mathit{Group}$ could therefore also be referred to as a *coherent system of automorphisms* of G.

Motivation

Given an element $s \in G$, we can define a corresponding coherent system of automorphisms $\hat{s}$ of G.

- The required automorphism $\hat{s}_{id_G} : G \xrightarrow{\sim} G$ is given by

  $$\hat{s}_{id_G}(g) = sgs^{-1}$$

  for all $g \in G$ (i.e. $\hat{s}_{id_G}$ is the inner automorphism of G induced by s).

- $\forall$ morphism $f : G \to H$ in $Group$, the required group automorphism $\hat{s}_f : H \xrightarrow{\sim} H$ is given by

  $$\hat{s}_f(h) = f(s)hf(s)^{-1}$$

  for all $h \in H$ (i.e. $\hat{s}_f$ is the inner automorphism of H induced by $f(s) \in H$).

Motivation

Now let $Aut(U_G)$ be the group of all automorphisms of $U_G : G \downarrow Group \to Group$, i.e. the group of all coherent systems of automorphisms of G.

Then we have a function (and in fact, a group homomorphism)

$$\hat{} : G \to Aut(U_G).$$

Bergman's Theorem

In his paper 'An inner automorphism is only an inner automorphism, but an inner endomorphism can be something strange', George Bergman then proved the following result:

Theorem (Bergman)

The map

$$\hat{} : G \to Aut(U_G).$$

is in fact a group **isomorphism**.

It follows that a group automorphism $f : G \xrightarrow{\sim} G$ *is* **inner** *iff there is a coherent system of automorphisms* α *of* G *such that*

$$\alpha_{id_G} = f.$$

Isotropy Groups of Algebraic Theories

Now let $\mathbb{T}$ be an algebraic theory, i.e. a mathematical theory whose axioms are just equations between terms (over a certain language). E.g. the theories of semigroups, monoids, (abelian) groups, (commutative) rings (with unit), lattices, and Boolean algebras are all algebraic theories.

There is a category $\mathbb{T}$-*mod* of all (set-based) $\mathbb{T}$-models and homomorphisms. For example, if $\mathbb{T}$ is the theory of groups, then $\mathbb{T}\text{-}mod = Group$.

Given any $\mathbb{T}$-model $M \in \mathbb{T}\text{-}mod$, there is a forgetful functor

$$U_M : M \downarrow \mathbb{T}\text{-}mod \to \mathbb{T}\text{-}mod,$$

defined as in the case for groups.

As in the case for groups, we can then consider the automorphism group of this functor U_M, or equivalently the group of all coherent systems of automorphisms of M.

Then we define the *isotropy group* of the $\mathbb{T}$-model M to be the group

$$Aut(U_M)$$

of all coherent systems of automorphisms of M.

This gives us a functor

$$\mathcal{Z}_{\mathbb{T}} : \mathbb{T}\text{-}mod \to Group,$$

which we call the *isotropy group (functor)* of the algebraic theory $\mathbb{T}$.

We can then turn (a consequence of) Bergman's theorem into a definition, as follows:

Definition

Let $\mathbb{T}$ be an algebraic theory with $M \in \mathbb{T}\text{-}mod$. Then we say that a $\mathbb{T}$-automorphism $f : M \xrightarrow{\sim} M$ is an **inner automorphism** if there is a coherent system of automorphisms $\alpha \in \mathcal{Z}_{\mathbb{T}}(M) = Aut(U_M)$ such that

$$\alpha_{id_M} = f.$$

Syntactic Characterization

So we can define a $\mathbb{T}$-automorphism $f : M \xrightarrow{\sim} M$ to be *inner* if it can be associated with a coherent system of automorphisms of M.

This is a purely *categorical* definition. Is it possible to characterize the inner automorphisms of a $\mathbb{T}$-model M using instead the syntax and axioms of the theory $\mathbb{T}$?

Syntactic Characterization

It turns out that we CAN. To do this, we take inspiration from some techniques used in the proof of Bergman's theorem.

First, if $M \in \mathbb{T}\text{-}mod$, then $M\langle x \rangle$ is the $\mathbb{T}$-model gained by adjoining an indeterminate x to M.

The elements $[t(x)] \in M\langle x \rangle$ are (congruence classes of) words $t(x)$ in x, the elements of M, and the function symbols of $\mathbb{T}$, modulo the equations of $\mathbb{T}$ and relations among elements of M.

If $[t(x)]$ is any element of $M\langle x \rangle$, then $[t(x)]$ induces a function

$$[t(x)]^M : M \to M,$$

defined by substitution into the indeterminate x.

Syntactic Characterization

Definition

Let $\mathbb{T}$ be an algebraic theory with $M \in \mathbb{T}$-*mod*. Let $M^{hom}\langle x \rangle$ be the set of all $[t(x)] \in M\langle x \rangle$ such that:

1. $[t(x)]$ is invertible, in the sense that there is some $[s(x)] \in M\langle x \rangle$ such that $[t[s/x]] = [x] = [s[t/x]]$.
2. For any n-ary function symbol f of $\mathbb{T}$ ($n \geq 0$) and indeterminates $x_1, \ldots, x_n$, the equality

$$[f(t(x_1), \ldots, t(x_n))] = [t[f(x_1, \ldots, x_n)/x]]$$

holds in the $\mathbb{T}$-model $M\langle x_1, \ldots, x_n \rangle$. We say that $[t(x)]$ **commutes generically** with all operations of $\mathbb{T}$.

Syntactic Characterization

Now we have the following syntactic description of the isotropy group of an algebraic theory $\mathbb{T}$:

Theorem

Let $\mathbb{T}$ be an algebraic theory. If $\mathcal{Z}_{\mathbb{T}}$ is the isotropy group functor of $\mathbb{T}$, then $\forall M \in \mathbb{T}$-mod we have

$$\mathcal{Z}_{\mathbb{T}}(M) = Aut(U_M) \cong M^{hom}\langle x \rangle,$$

natural in M.

As a consequence, a function $f : M \to M$ is an inner automorphism of M iff there is some $[t(x)] \in M^{hom}\langle x \rangle$ such that

$$f = [t(x)]^M : M \to M.$$

Examples

With this concrete characterization of the isotropy group of an algebraic theory $\mathbb{T}$ at hand, we can now (more easily) compute the isotropy groups of several algebraic theories $\mathbb{T}$:

- If $\mathbb{T}$ has no axioms, then the isotropy group of $\mathbb{T}$ is trivial, i.e. $\forall M \in \mathbb{T}\text{-}mod$ we have $\mathcal{Z}_{\mathbb{T}}(M) = \{[x]\} \cong 1$, the trivial group.
- If $\mathbb{T}$ is the theory of groups, then Bergman essentially proved that $\forall G \in Group$ we have

$$\mathcal{Z}_{\mathbb{T}}(G) = \{[gxg^{-1}] \in G\langle x \rangle \mid g \in G\} \cong G.$$

- If $\mathbb{T}$ is the theory of monoids, then $\forall M \in Monoid$ we have

$$\mathcal{Z}_{\mathbb{T}}(M) = \{[mxm'] \in M\langle x \rangle \mid m \text{ is invertible in } M \text{ and } m' = m^{-1}\}.$$

Examples

- If $\mathbb{T}$ is the theory of abelian groups, then $\forall G \in \mathit{AbGroup}$ we have

$$\mathcal{Z}_{\mathbb{T}}(G) = \{[x], [-x]\} \cong \mathbb{Z}_2.$$

- If $\mathbb{T}$ is the theory of commutative monoids, then the isotropy group of $\mathbb{T}$ is trivial.
- If $\mathbb{T}$ is the theory of (non-commutative) rings with 1, then $\forall R \in \mathit{Ring}$ we have

$$\mathcal{Z}_{\mathbb{T}}(R) = \{[rxr^{-1}] \in R\langle x\rangle \mid \ r \text{ is a unit } \}.$$

- If $\mathbb{T}$ is the theory of commutative rings with 1, then the isotropy group of $\mathbb{T}$ is trivial.
- If $\mathbb{T}$ is the theory of lattices, then the isotropy group of $\mathbb{T}$ is trivial.

Connection to topos theory

- For any category $\mathbb{C}$ we can define the isotropy functor $\mathcal{Z} = \mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to Group$ so that

$$C \mapsto Aut(\mathbb{C}/C \to \mathbb{C})$$

  $\forall C \in \mathbb{C}$.
- When $\mathcal{E}$ is a Grothendieck topos, the isotropy functor $\mathcal{Z} : \mathcal{E}^{op} \to Group$ is representable by a group object $Z_{\mathcal{E}}$ internal to $\mathcal{E}$.
- When $\mathcal{E} = Set^{\mathbb{C}^{op}}$ for some category $\mathbb{C}$, then the group object $Z_{\mathcal{E}} \in Set^{\mathbb{C}^{op}}$ is exactly the isotropy functor

$$\mathcal{Z}_{\mathbb{C}} : \mathbb{C}^{op} \to Group \hookrightarrow Set.$$

- In this talk, we have been considering the special case where $\mathbb{C} = \mathbb{T}\text{-}mod^{op}$ for an algebraic theory $\mathbb{T}$.

Thank you!